Strong convergence rates for an explicit numerical approximation method for stochastic evolution equations with non-globally Lipschitz continuous nonlinearities

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Abstract

In this article we propose a new, explicit and easily implementable numerical method for approximating a class of semilinear stochastic evolution equations with non-globally Lipschitz continuous nonlinearities. We establish strong convergence rates for this approximation method in the case of semilinear stochastic evolution equations with globally monotone coefficients. Our strong convergence result, in particular, applies to a class of stochastic reaction-diffusion partial differential equations.

Contents

1 Introduction

In this article we are interested in strong approximations of stochastic evolution equations (SEEs) with non-globally Lipschitz continuous nonlinearities. In the literature, there are nowadays a number of strong approximation results for such stochastic evolution equations on finite dimensional state spaces, that is, for finite dimensional stochastic ordinary differential equations (SODEs). For example, Theorem 2.1 in Hutzenthaler et al. [15] (see also Theorem 2.1 in Hutzenthaler et al. [13]) proves that the classical explicit Euler scheme (also known as Euler-scheme or Euler-Maruyama scheme; see Maruyama [24]) diverges strongly and numerically weakly in finite time time when applied to a SODE with superlinearly growing (and hence non-globally Lipschitz continuous) nonlinearities. Theorem 2.4 in Hu [10] establishes that the drift-implicit Euler scheme (also known as Backward Euler scheme or implicit Euler scheme) overcomes this lack of strong convergence of the explicit Euler scheme and converges with the usual strong order $1/2$ to the solution process in the case of some SODEs with non-globally Lipschitz continuous but globally monotone coefficients. However, the drift-implicit Euler scheme can often only be realized approximatively and this approximation of the drift-implicit Euler scheme is computationally more expensive than the explicit Euler scheme, particularly when the state space of the considered SEE is high dimensional (see, e.g., Figure 4 in Hutzenthaler et al. [14]), because the solution of a nonlinear equation has to be computed approximatively at each time step. In Hutzenthaler et al. [14] a modified version of the explicit Euler scheme, which is explicit and easy to implement, has been proposed and shown to converge with the usual strong order $1/2$ to the solution process in the case of some SODEs with non-globally Lipschitz continuous but globally monotone coefficients. The above mentioned articles contain just a few selected illustrative results and a number of further and partially significantly improved strong approximation results for SODEs with non-globally Lipschitz continuous nonlinearities are available in the literature; see, e.g., [2], [11], [12], [20], [26], [27], [28], [29], [31], [32], and the references mentioned in

the above named references for some strong numerical approximations results for explicit schemes and multi-dimensional SODEs with non-globally Lipschitz continuous coefficients. At least parts of the above outlined story has already been extended to SEEs on infinite dimensional state spaces including stochastic partial differential equations (SPDEs) as special cases. In particular, it is clear that Theorem 2.1 in Hutzenthaler et al. [15] also extends to some SEEs with superlinearly growing nonlinearities on infinite dimensional state space (see Section 5.1 in Kurniawan [22]). More specifically, the explicit, the exponential, and the linear-implicit Euler method are known to diverge in the strong and numerically weak sense in the case of some SPDEs with superlinearly growing coefficients. Moreover, strong convergence but with no rate of convergence of an full-discrete drift-implicit Euler method has, e.g., been proven in Theorem 2.10 in Gyöngy & Millet [7] in the case of some SEEs with non-globally Lipschitz continuous nonlinearities; see also, e.g., Theorem 7.1 in Brzeźniak et al. [3] and Theorem 5.4 in Kovács et al. [21]. Furthermore, in Gyöngy et al. [8], in Hutzenthaler & Jentzen [12, equation (3.145)], and in Kurniawan [22] appropriately modified, explicit and easily realizable versions of the explicit, the exponential and the linear-implicit Euler scheme have been considered for approximating semilinear SEEs with non-globally Lipschitz continuous nonlinearities. In addition, in Gyöngy et al. [8] and in Kurniawan [22], it has also been proved that the considered approximation methods converge strongly to the solution processes of the investigated SEEs. The results in Gyöngy et al. [8] and in Kurniawan [22] do not prove any rate of strong convergence. In this article we propose a modified variant of the scheme considered in Kurniawan [22, Section 2] and prove for every $p \in (0, \infty)$ that this scheme convergences in strong L^p-distance with an appropriate strong rate of convergence in the case of a class of semilinear SEEs with non-globally Lipschitz continuous but globally monotone nonlinearities; see Theorem 7.6 (the main result of this article) in Section 7 below for details. To the best of our knowledge, Theorem 7.6 below is the first result in the literature which establishes a strong convergence rate for an explicit and easily implementable full-discrete numerical approximation method for semilinear SPDEs with non-globally Lipschitz continuous nonlinearities.

In the remainder of this introductory section we illustrate Theorem 7.6 by presenting a consequence of it in Theorem 1.1 below. For this we consider the following setting (see Section 7 below for our general framework). Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a countable orthonormal basis of H, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $\lambda \colon \mathbb{H} \to \mathbb{R}$ be a function satisfying $\sup_{h\in\mathbb{H}} \lambda_h < 0$, let $A \colon D(A) \subseteq H \to H$ be the linear operator such that $D(A) = \big\{v \in H \colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty\big\}$ and such that for all $v \in D(A)$ it holds that $Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (see, e.g., Theorem and Definition 2.5.32 in [16]), let $T \in (0,\infty)$, $c \in [1,\infty)$, $\gamma \in [0, 1/2)$, $\alpha \in [0, 1-\gamma)$, $\beta \in [0, 1/2-\gamma)$, $\delta \in [0,\gamma]$, $\xi \in H_{1/2}$, $\theta \in (0, 1/4]$, $p \in [2,\infty)$, $\kappa \in (2/p, \infty)$, $\varepsilon \in (0,\infty)$, $F \in \mathcal{C}(H_\gamma, H)$, $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U,H))$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_U-Wiener process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, let $X \colon [0,T] \times \Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s)\, ds + \int_0^t e^{(t-s)A} B(X_s)\, dW_s, \tag{1}$$

let $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$ and $(\hat{P}_J)_{J\in\mathcal{P}(\mathbb{U})} \subseteq L(U)$ be the linear operators with the property that[1] for all $x \in H$, $y \in U$, $I \in \mathcal{P}(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$ it holds that $P_I(x) = \sum_{h\in I}\langle h, x\rangle_H h$ and $\hat{P}_J(y) = \sum_{u\in J}\langle u, y\rangle_U u$,

[1] Here and below we denote for a set S by $\mathcal{P}(S)$ the power set of S and we denote for a set S by $\mathcal{P}_0(S)$ the set given by $\mathcal{P}_0(S) = \{M \in \mathcal{P}(S) \colon M \text{ is a finite set}\}$.

let $Y^{N,I,J}\colon [0,T]\times\Omega\to H_\gamma$, $N\in\mathbb{N}$, $I\in\mathcal{P}(\mathbb{H})$, $J\in\mathcal{P}(\mathbb{U})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes such that for all $t\in[0,T]$, $N\in\mathbb{N}$, $I\in\mathcal{P}(\mathbb{H})$, $J\in\mathcal{P}(\mathbb{U})$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} Y_t^{N,I,J} &= e^{tA}P_I\xi + \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\,\mathbb{1}_{\left\{\|P_I F(Y^{N,I,J}_{\lfloor s\rfloor_{T/N}})\|_H+\|P_I B(Y^{N,I,J}_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)}\leq(\frac{N}{T})^\theta\right\}}\,P_I F(Y^{N,I,J}_{\lfloor s\rfloor_{T/N}})\,ds \\ &\quad + \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\,\mathbb{1}_{\left\{\|P_I F(Y^{N,I,J}_{\lfloor s\rfloor_{T/N}})\|_H+\|P_I B(Y^{N,I,J}_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)}\leq(\frac{N}{T})^\theta\right\}}\,P_I B(Y^{N,I,J}_{\lfloor s\rfloor_{T/N}})\hat{P}_J\,dW_s, \end{aligned} \tag{2}$$

and assume that for all $x,y\in H_\gamma$, $v,w\in H_1$ it holds that $\langle v,F(v)\rangle_H+\frac{2c(c+1)p\max\{\kappa,1/\theta\}-1}{2}\|B(v)\|^2_{\mathrm{HS}(U,H)}\leq c\,(1+\|v\|_H^2)$, $\max\{\|F(x)\|_{H_{-\alpha}},\|B(x)\|_{\mathrm{HS}(U,H_{-\beta})}\}\leq c\,(1+\|x\|_H^c)$, $\langle v-w,Av-Aw+F(v)-F(w)\rangle_H+\frac{(p-1)(1+\varepsilon)}{2}\|B(v)-B(w)\|^2_{\mathrm{HS}(U,H)}\leq c\,\|v-w\|_H^2$, and $\max\{\|F(x)-F(y)\|_H,\|B(x)-B(y)\|_{\mathrm{HS}(U,H)}\}\leq c\,\|x-y\|_{H_\delta}(1+\|x\|^c_{H_\gamma}+\|y\|^c_{H_\gamma})$. In the following we refer to the numerical approximations in (2) as nonlinearities-stopped exponential Euler approximations (cf., e.g., Kurniawan [22, Section 2]).

Theorem 1.1. *Assume the setting in the second paragraph of Section 1. Then for every $\eta\in[0,1/2)$ there exists a real number $K\in[0,\infty)$ such that for all $N\in\mathbb{N}$, $I\in\mathcal{P}_0(\mathbb{H})$, $J\in\mathcal{P}_0(\mathbb{U})$ it holds that*

$$\sup_{t\in[0,T]}\big\|X_t-Y_t^{N,I,J}\big\|_{L^p(\mathbb{P};H)}\leq K\left(N^{\delta-\eta}+\Big|\sup_{h\in\mathbb{H}\setminus I}\lambda_h\Big|^{(\delta-\eta)}+\sup_{v\in H_\eta}\left[\tfrac{\|B(v)\hat{P}_{\mathbb{U}\setminus J}\|_{\mathrm{HS}(U,H_{-\eta})}}{(1+\|v\|_{H_\eta})^\kappa}\right]\right). \tag{3}$$

Theorem 1.1 is a direct consequence of Theorem 7.6 in Section 7 below. In the following we give an outline of the proof of Theorem 7.6 and we also sketch the content of the remaining sections of this article. The proof of Theorem 7.6 is divided into several pieces. First, in Section 2 we establish a priori moment estimates for the approximation scheme (2) in the H_0-norm. In Section 3 we use twice suitable bootstrap-type arguments to strengthen these a priori moment bounds in the H_0-norm to obtain for any $\eta\in(-\infty,1/2)$ a priori moment estimates for the approximation scheme (2) in the H_η-norm. In Section 4 we use the a priori moment bounds established in Sections 2 and 3 to estimate the temporal discretization errors of the nonlinearities-stopped exponential Euler approximations in (2); see Corollary 4.4 in Section 4. Our main idea in the proof of Corollary 4.4 is not to estimate the error of the numerical approximations (2) and the solution process X of the SEE (1) directly but instead to plug, similar as in Jentzen & Kurniawan [18, (11), (70), (136)], appropriate approximation processes, so-called semilinear integrated counterparts of (2), in between, to estimate the difference of the numerical approximations (2) and their semilinear integrated counterparts in a straightforward way (see Lemma 4.2) and to employ the perturbation estimate in Theorem 2.10 in Hutzenthaler & Jentzen [11] to estimate the differences of the solution process of the considered SEE and the semilinear integrated counterparts of the nonlinearities-stopped exponential Euler approximations (see Lemma 4.3). Combining Lemma 4.2 and Lemma 4.3 with the triangle inequality will then immediately result in Corollary 4.4. In Section 5 and Section 6 we establish an auxiliary spatial approximation result (see Proposition 6.4 in Section 6) which we use in Section 7 to prove Theorem 7.6. In addition, we use consequences of the perturbation estimate in Theorem 2.10 in Hutzenthaler & Jentzen [11] to establish strong convergence rates for spatial spectral Galerkin approximations (see Lemma 7.1) and for noise approximations (see Lemma 7.2) of the considered SEEs. Combining the spatial approximation result in Lemma 7.1 in Section 7 and the noise approximation result in Lemma 7.2 in Section 7 with the results established in the earlier sections of this article (especially the temporal approximation result in Corollary 4.4 in Section 4) will then allow us to complete the proof of Theorem 7.6 in Section 7. In Section 8 we illustrate the consequences of Theorem 7.6 and Theorem 1.1 respectively in the case of an illustrative example SPDE.

More formally, suppose that $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ is the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0,1)$ to $\mathbb{R}$, let $\rho \in (0,\infty)$, $(r_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, and $(e_n)_{n\in\mathbb{N}} \subseteq H$ satisfy that $\mathbb{H} = \{e_1, e_2, \dots\}$, that $\sup_{n\in\mathbb{N}}(n\,|r_n|) < \infty$, and that[2] for all $n \in \mathbb{N}$ and $\mu_{(0,1)}$-almost all $x \in (0,1)$ it holds that $e_n(x) = \sqrt{2}\sin(n\pi x)$, $\lambda_{e_n} = -n^2\pi^2$ and $\xi(x) \geq 0$, let $Q \in L_1(H)$ satisfy that for all $v \in H$ it holds that $Qv = \sum_{n=1}^{\infty} r_n \langle e_n, v\rangle_H e_n$, assume that $\gamma \in (1/4, 1/2)$, and assume that for all $v \in H_\gamma$, $u \in H$ and $\mu_{(0,1)}$-almost all $x \in (0,1)$ it holds that $\big(F(v)\big)(x) = |v(x)|\,(\rho - v(x))$ and $\big(B(v)u\big)(x) = v(x)\cdot(Q^{1/2}u)(x)$. The stochastic process X is thus a solution process of the stochastic reaction-diffusion partial differential equation

$$dX_t(x) = \Big[\tfrac{\partial^2}{\partial x^2}X_t(x) + X_t(x)\,(\rho - X_t(x))\Big]\,dt + \sigma X_t(x)\,dW_t(x),\ X_0(x) = \xi(x),\ X_t(0) = X_t(1) = 0$$

for $t \in [0,T]$, $x \in (0,1)$. Then we show in Section 8 that Theorem 1.1 ensures that for every $q, \iota \in (0,\infty)$ there exists a real number $K \in [0,\infty)$ such that for all $N, n, m \in \mathbb{N}$ it holds that

$$\sup_{t\in[0,T]} \big\|X_t - Y_t^{N,\{e_1,e_2,\dots,e_n\},\{e_1,e_2,\dots,e_m\}}\big\|_{L^q(\mathbb{P};H)} \leq K\left(\frac{1}{N^{(1/2-\iota)}} + \frac{1}{n^{(1-\iota)}} + \frac{1}{m^{(1-\iota)}}\right). \tag{4}$$

In particular, this shows that for every $q, \iota \in (0,\infty)$ there exists a real number $K \in [0,\infty)$ such that for all $n \in \mathbb{N}$ it holds that $\sup_{t\in[0,T]} \|X_t - Y_t^{n^2,\{e_1,e_2,\dots,e_n\},\{e_1,e_2,\dots,e_n\}}\|_{L^q(\mathbb{P};H)} \leq K\cdot n^{(\iota-1)}$.

1.1 Notation

Throughout this article the following notation is used. For a set S we denote by $\mathrm{Id}_S\colon S \to S$ the identity mapping on S, that is, it holds for all $x \in S$ that $\mathrm{Id}_S(x) = x$. For a set A we denote by $\mathcal{P}(A)$ its power set, we denote by $|A| \in \{0,1,2,\dots\}\cup\{\infty\}$ the number of elements of A, and we denote by $\mathcal{P}_0(A)$ the set given by $\mathcal{P}_0(A) = \{B \in \mathcal{P}(A)\colon |B| < \infty\}$. For measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ we denote by $\mathcal{M}(\mathcal{F}_1, \mathcal{F}_2)$ the set of all $\mathcal{F}_1/\mathcal{F}_2$-measurable functions. For topological spaces (X, τ_X) and (Y, τ_Y) we denote by $\mathcal{C}(X,Y)$ the set of all continuous functions from X to Y. For a topological space (X,τ) we denote by $\mathcal{B}(X)$ the sigma-algebra of all Borel measurable sets in X. Let $\lfloor\cdot\rfloor_h\colon \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, be the mappings with the property that for all $h \in (0,\infty)$, $t \in \mathbb{R}$ it holds that $\lfloor t\rfloor_h = \max\big\{(-\infty, t] \cap \{0, h, -h, 2h, -2h, \dots\}\big\}$. For a natural number $d \in \mathbb{N}$ and a Borel measurable set $A \in \mathcal{B}(\mathbb{R}^d)$ we denote by $\mu_A\colon \mathcal{B}(A) \to [0,\infty]$ the Lebesgue-Borel measure on A. For a normed $\mathbb{R}$-vector space $(V, \|\cdot\|_V)$ and a real number $\rho \in [0,\infty)$ we denote by $L^\rho(V)$ the set given by $L^\rho(V) = \big\{A \in L(V)\colon \|A\|_{L(V)} \leq \rho\big\}$.

1.2 Setting

Throughout this article the following setting is frequently used. Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$, $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function satisfying $\sup_{h\in\mathbb{H}} \lambda_h < 0$, let $A\colon D(A) \subseteq H \to H$ be a linear operator such that $D(A) = \big\{v \in H\colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty\big\}$ and such that for all $v \in D(A)$ it holds that $Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (see, e.g., Theorem and Definition 2.5.32 in [16]), let $\alpha, \beta, \gamma \in [0,\infty)$, $T \in (0,\infty)$, $a, c, C \in [1,\infty)$, $\delta \in [0,\gamma]$, $\theta \in (0, \frac{1}{4}]$, $p \in [2,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $\xi \in \mathcal{M}(\mathcal{F}_0, \mathcal{B}(H_\gamma))$, and let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_U-Wiener process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$.

[2]Here and below we denote for a natural number $d \in \mathbb{N}$ and a Borel measurable set $B \in \mathcal{B}(\mathbb{R}^d)$ by $\mu_B\colon \mathcal{B}(B) \to [0,\infty]$ the Lebesgue-Borel measure on B.

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2 Strong a priori moment bounds for nonlinearities-stopped schemes

In this section we establish strong a priori moment bounds for a class of nonlinearities-stopped exponential Euler approximations; see Lemma 2.2 below for the main result of Section 2. Related arguments/results can, e.g., be found in Section 2 in Kurniawan [22] and in Section 3 in Gyöngy et al. [8].

2.1 Setting

Assume the setting in Section 1.2, let $F \in \mathcal{M}\big(\mathcal{B}(H_\gamma), \mathcal{B}(H)\big), B \in \mathcal{M}\big(\mathcal{B}(H_\gamma), \mathcal{B}(\mathrm{HS}(U,H))\big)$, $N \in \mathbb{N}, K \in [0,\infty)$ satisfy $K = 3(p-2) + 2C^{p/2} + 2\big[T^{1-2\theta} + \frac{p}{2}T^{\frac{1}{2}-2\theta}\big]^{p/2}$, let $Y\colon [0,T]\times\Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} Y_t = e^{tA}\xi &+ \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\, \mathbb{1}_{\left\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} F(Y_{\lfloor s\rfloor_{T/N}})\, ds \\ &+ \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\, \mathbb{1}_{\left\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} B(Y_{\lfloor s\rfloor_{T/N}})\, dW_s, \end{aligned} \tag{5}$$

and assume that for all $x \in H_\gamma$ it holds that $\langle x, F(x)\rangle_H + \frac{p-1}{2}\|B(x)\|^2_{\mathrm{HS}(U,H)} \leq C\,(1 + \|x\|_H^2)$.

2.2 Strong a priori moment bounds for nonlinearities-stopped schemes

lemma 2.1. *Assume the setting in Section 2.1 and let $x \in H, h \in (0,T], t \in [0,h]$. Then*

$$\mathbb{E}\bigg[\Big\|e^{tA}\Big(x + \mathbb{1}_{\{\|F(x)\|_H + \|B(x)\|_{HS(U,H)} \leq h^{-\theta}\}}\, \big[tF(x) + \textstyle\int_0^t B(x)\, dW_s\big]\Big)\Big\|_H^p\bigg] \leq e^{Kt}\,(\|x\|_H^p + Kt)\,. \tag{6}$$

Proof of Lemma 2.1. W.l.o.g. we assume that $\|F(x)\|_H + \|B(x)\|_{\mathrm{HS}(U,H)} \leq h^{-\theta}$ (otherwise (6) is clear). Let $Y^x\colon [0,T]\times\Omega \to H, x \in H$, be stochastic processes such that for all $t \in [0,T], x \in H$ it holds $\mathbb{P}$-a.s. that $Y_t^x = x + \int_0^t F(x)\, ds + \int_0^t B(x)\, dW_s$ and let $f\colon H \to \mathbb{R}$ be the function with the property that for all $x \in H$ it holds that $f(x) = \|x\|_H^p$. Then f is twice continuously differentiable and for all $x, v, w \in H$ it holds that

$$\begin{aligned} f'(x)(v) &= p\,\|x\|_H^{p-2}\,\langle x, v\rangle_H, \\ f''(x)(v,w) &= \begin{cases} p\,\|x\|_H^{p-2}\,\langle v, w\rangle_H + p(p-2)\,\|x\|_H^{p-4}\,\langle x, v\rangle_H\langle x, w\rangle_H, & x \neq 0 \\ p\,\|x\|_H^{p-2}\,\langle v, w\rangle_H, & x = 0. \end{cases} \end{aligned} \tag{7}$$

Itô's formula hence proves that it holds $\mathbb{P}$-a.s. that

$$
\begin{aligned}
\|Y_t^x\|_H^p &= \|Y_0^x\|_H^p + \int_0^t f'(Y_s^x) F(x)\, ds + \tfrac{1}{2} \sum_{u \in \mathbb{U}} f''(Y_s^x)(B(x)u, B(x)u)\, ds + \int_0^t f'(Y_s^x) B(x)\, dW_s \\
&= \|Y_0^x\|_H^p + \int_0^t p\, \|Y_s^x\|_H^{p-2} \langle Y_s^x, F(x)\rangle_H\, ds + \int_0^t p\, \|Y_s^x\|_H^{p-2} \langle Y_s^x, B(x)\, dW_s\rangle_H \\
&\quad + \tfrac{1}{2} \int_0^t \sum_{u \in \mathbb{U}} \Big[p\, \|Y_s^x\|_H^{p-2} \langle B(x)u, B(x)u\rangle_H + p(p-2)\, \mathbb{1}_{\{Y_s^x \neq 0\}}\, \|Y_s^x\|_H^{p-4}\, |\langle Y_s^x, B(x)u\rangle_H|^2 \Big]\, ds.
\end{aligned} \tag{8}
$$

The triangle inequality, Fubini's theorem, and the Hölder inequality therefore show that

$$
\begin{aligned}
&\mathbb{E}[\|Y_t^x\|_H^p] \\
&\le \mathbb{E}[\|Y_0^x\|_H^p] + p \int_0^t \mathbb{E}\big[\|Y_s^x\|_H^{p-2} \langle Y_s^x, F(x)\rangle_H\big]\, ds + \tfrac{p(p-1)}{2} \int_0^t \mathbb{E}\Big[\|Y_s^x\|_H^{p-2} \textstyle\sum_{u \in \mathbb{U}} \|B(x)u\|_H^2\Big]\, ds \\
&= \|x\|_H^p + p \Big[\langle x, F(x)\rangle_H + \tfrac{p-1}{2} \|B(x)\|_{\mathrm{HS}(U,H)}^2\Big] \int_0^t \mathbb{E}\big[\|Y_s^x\|_H^{p-2}\big]\, ds \\
&\quad + p \int_0^t \mathbb{E}\big[\|Y_s^x\|_H^{p-2} \langle F(x)s + \textstyle\int_0^s B(x)\, dW_r, F(x)\rangle_H\big]\, ds \\
&\le \|x\|_H^p + p \Big[\langle x, F(x)\rangle_H + \tfrac{p-1}{2} \|B(x)\|_{\mathrm{HS}(U,H)}^2\Big]^+ \int_0^t \|Y_s^x\|_{L^p(\mathbb{P};H)}^{p-2}\, ds \\
&\quad + p \int_0^t \|Y_s^x\|_{L^p(\mathbb{P};H)}^{p-2} \|\langle F(x)s + \textstyle\int_0^s B(x)\, dW_r, F(x)\rangle_H\|_{L^{p/2}(\mathbb{P};H)}\, ds.
\end{aligned} \tag{9}
$$

In the next step we apply the Cauchy-Schwarz inequality and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [5] to (9) to obtain that

$$
\begin{aligned}
\mathbb{E}[\|Y_t^x\|_H^p] &\le \|x\|_H^p + pC\,[1 + \|x\|_H^2] \int_0^t \|Y_s^x\|_{L^p(\mathbb{P};H)}^{p-2}\, ds \\
&\quad + p \int_0^t \|Y_s^x\|_{L^p(\mathbb{P};H)}^{p-2} \Big[\|F(x)\|_H^2\, s + \sqrt{s}\tfrac{p}{2}\, \|B(x)\|_{\mathrm{HS}(U,H)} \|F(x)\|_H\Big]\, ds.
\end{aligned} \tag{10}
$$

Young's inequality hence proves that

$$
\begin{aligned}
\mathbb{E}[\|Y_t^x\|_H^p] &\le \|x\|_H^p + p\int_0^t \left[\tfrac{2}{p}C^{\frac{p}{2}} + \tfrac{p-2}{p}\|Y_s^x\|_{L^p(\mathbb{P};H)}^p\right] ds + p\int_0^t \left[\tfrac{2}{p}C^{\frac{p}{2}}\|x\|_H^p + \tfrac{p-2}{p}\|Y_s^x\|_{L^p(\mathbb{P};H)}^p\right] ds \\
&\quad + p\int_0^t \tfrac{p-2}{p}\|Y_s^x\|_{L^p(\mathbb{P};H)}^p + \tfrac{2}{p}\left[\|F(x)\|_H^2 s + \sqrt{s}\tfrac{p}{2}\|B(x)\|_{\mathrm{HS}(U,H)}\|F(x)\|_H\right]^{\frac{p}{2}} ds \\
&= \|x\|_H^p + \int_0^t 2C^{\frac{p}{2}} + (p-2)\|Y_s^x\|_{L^p(\mathbb{P};H)}^p\, ds + \int_0^t 2\|x\|_H^p C^{\frac{p}{2}} + (p-2)\|Y_s^x\|_{L^p(\mathbb{P};H)}^p\, ds \\
&\quad + \int_0^t (p-2)\|Y_s^x\|_{L^p(\mathbb{P};H)}^p + 2\left[\|F(x)\|_H^2 s + \sqrt{s}\tfrac{p}{2}\|B(x)\|_{\mathrm{HS}(U,H)}\|F(x)\|_H\right]^{\frac{p}{2}} ds \qquad (11) \\
&= \|x\|_H^p + 3(p-2)\int_0^t \|Y_s^x\|_{L^p(\mathbb{P};H)}^p\, ds \\
&\quad + \int_0^t 2C^{\frac{p}{2}}(1+\|x\|_H^p) + 2\left[\|F(x)\|_H^2 s + \sqrt{s}\tfrac{p}{2}\|B(x)\|_{\mathrm{HS}(U,H)}\|F(x)\|_H\right]^{\frac{p}{2}} ds \\
&\le \|x\|_H^p + 3(p-2)\int_0^t \|Y_s^x\|_{L^p(\mathbb{P};H)}^p\, ds + \left(2C^{\frac{p}{2}}\left[1+\|x\|_H^p\right] + 2\left[t^{1-2\theta} + \tfrac{p}{2}t^{\frac{1}{2}-2\theta}\right]^{\frac{p}{2}}\right) t.
\end{aligned}
$$

Gronwall's lemma therefore shows that

$$
\begin{aligned}
\mathbb{E}[\|Y_t^x\|_H^p] &\le e^{3(p-2)t}\left[\|x\|_H^p + t\left(2C^{\frac{p}{2}}\left[1+\|x\|_H^p\right] + 2\big[T^{1-2\theta} + \tfrac{p}{2}T^{\frac{1}{2}-2\theta}\big]^{\frac{p}{2}}\right)\right] \\
&= e^{3(p-2)t}\left(\left(1+2tC^{p/2}\right)\|x\|_H^p + t\left[2C^{\frac{p}{2}} + 2\big[T^{1-2\theta} + \tfrac{p}{2}T^{\frac{1}{2}-2\theta}\big]^{\frac{p}{2}}\right]\right).
\end{aligned} \qquad (12)
$$

Note that for all $t \in [0,T]$ it holds that $1 + 2tC^{p/2} \le e^{2tC^{p/2}}$. Combining this with (12) shows that for all $t \in [0,T]$ it holds that

$$
\mathbb{E}[\|Y_t^x\|_H^p] \le e^{(3(p-2)+2C^{p/2})t}\left(\|x\|_H^p + t\left(2C^{p/2} + 2\big[T^{1-2\theta} + \tfrac{p}{2}T^{\frac{1}{2}-2\theta}\big]^{\frac{p}{2}}\right)\right). \qquad (13)
$$

The proof of Lemma 2.1 is thus completed. □

lemma 2.2. *Assume the setting in Section 2.1 and let $t \in [0,T]$. Then*

$$
\mathbb{E}[\|Y_t\|_H^p] \le \left(\mathbb{E}[\|Y_0\|_H^p] + Kt\right) e^{Kt}. \qquad (14)
$$

Proof of Lemma 2.2. Lemma 2.1 implies that for all $n \in \{1, 2, \dots, N\}$ it holds that

$$
\begin{aligned}
&\mathbb{E}\Big[\big\|Y_{\frac{n}{N}T}\big\|_H^p\Big] \le e^{KT/N}\mathbb{E}\Big(\Big[\big\|Y_{\frac{n-1}{N}T}\big\|_H^p\Big] + K\tfrac{T}{N}\Big) \\
&\le e^{KT/N}\left(\left(e^{KT/N}\mathbb{E}\Big[\big\|Y_{\frac{n-2}{N}T}\big\|_H^p\Big] + K\tfrac{T}{N}\right) + K\tfrac{T}{N}\right) \le \ldots \le \mathbb{E}[\|Y_0\|_H^p]\, e^{TKn/N} + K\tfrac{T}{N}\sum_{j=1}^{n} e^{TKj/N}.
\end{aligned} \qquad (15)
$$

Again Lemma 2.1 hence proves that

$$
\begin{aligned}
\mathbb{E}[\|Y_t\|_H^p] &= \mathbb{E}\Bigg[\Big\|e^{(t-\lfloor t\rfloor_{T/N})A}\Big(Y_{\lfloor t\rfloor_{T/N}} + \mathbb{1}_{\left\{\|F(Y_{\lfloor t\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor t\rfloor_{T/N}})\|_{HS(U,H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} \\
&\quad \cdot \left[(t-\lfloor t\rfloor_{T/N})F(Y_{\lfloor t\rfloor_{T/N}}) + \textstyle\int_{\lfloor t\rfloor_{T/N}}^t B(Y_{\lfloor t\rfloor_{T/N}})\,dW_s\right]\Big)\Big\|_H^p\Bigg] \\
&\leq e^{K(t-\lfloor t\rfloor_{T/N})}\left(\mathbb{E}\left[\|Y_{\lfloor t\rfloor_{T/N}}\|_H^p\right] + K(t-\lfloor t\rfloor_{T/N})\right) \\
&\leq e^{K(t-\lfloor t\rfloor_{T/N})}\left[\mathbb{E}[\|Y_0\|_H^p]\,e^{K\lfloor t\rfloor_{T/N}} + K\tfrac{T}{N}\sum_{j=1}^{\lfloor t\rfloor_{T/N}\frac{N}{T}} e^{TKj/N} + K(t-\lfloor t\rfloor_{T/N})\right] \\
&\leq \mathbb{E}[\|Y_0\|_H^p]\,e^{Kt} + Kte^{Kt}.
\end{aligned}
\tag{16}
$$

The proof of Lemma 2.2 is thus completed. ☐

3 Strengthened strong a priori moment bounds based on bootstrap-type arguments

In this section we use the strong a priori moment bounds established in Section 2 to derive appropriately strengthened strong a priori moment bounds for numerical approximation processes and solution processes of SPDEs; see Lemma 3.1 and Lemma 3.2 below. The proofs of Lemma 3.1 and Lemma 3.2 are based on suitable bootstrap-type arguments. Bootstrap-type arguments of this kind have been intensively used in the literature to establish regularity properties of solutions of (stochastic) evolution equations.

3.1 Setting

Assume the setting in Section 1.2, let $F \in \mathcal{M}\big(\mathcal{B}(H_\gamma), \mathcal{B}(H)\big)$, $B \in \mathcal{M}\big(\mathcal{B}(H_\gamma), \mathcal{B}(\mathrm{HS}(U,H))\big)$, $\kappa \in \mathcal{M}\big(\mathcal{B}([0,T]), \mathcal{B}([0,T])\big)$ satisfy that for all $s \in [0,T]$ it holds that $\kappa(s) \leq s$, and let $Y, Z\colon [0,T] \times \Omega \to H_\gamma$ be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^t \|e^{(t-\kappa(s))A}F(Z_s)\|_H + \|e^{(t-\kappa(s))A}B(Z_s)\|_{\mathrm{HS}(U,H)}^2\,ds < \infty$ and

$$
Y_t = e^{tA}\xi + \int_0^t e^{(t-\kappa(s))A}F(Z_s)\,ds + \int_0^t e^{(t-\kappa(s))A}B(Z_s)\,dW_s. \tag{17}
$$

3.2 A first bootstrap-type argument for a priori bounds

lemma 3.1. *Assume the setting in Section 3.1, let $t \in [0,T]$, assume that $\gamma < \min\{1-\alpha, 1/2-\beta\}$, and assume that for all $x \in H_\gamma$ it holds that $\max\{\|F(x)\|_{H_{-\alpha}}, \|B(x)\|_{\mathrm{HS}(U,H_{-\beta})}\} \leq C(1+\|x\|_H^a)$. Then*

$$
\|Y_t\|_{L^p(\mathbb{P};H_\gamma)} \leq \|\xi\|_{L^p(\mathbb{P};H_\gamma)} + C\left[\tfrac{t^{1-(\gamma+\alpha)}}{1-(\gamma+\alpha)} + \sqrt{\tfrac{p(p-1)}{2(1-2(\gamma+\beta))}}\,t^{1/2-(\gamma+\beta)}\right]\left[1 + \sup\nolimits_{s\in[0,t]}\|Z_s\|_{L^{pa}(\mathbb{P};H)}^a\right]. \tag{18}
$$

Proof of Lemma 3.1. The triangle inequality and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [5] imply

$$\begin{aligned}\|Y_t\|_{L^p(\mathbb{P};H_\gamma)} \leq \|\xi\|_{L^p(\mathbb{P};H_\gamma)} + \int_0^t \left\|e^{(t-\kappa(s))A}F(Z_s)\right\|_{L^p(\mathbb{P};H_\gamma)} ds \\ + \left[\tfrac{p(p-1)}{2}\int_0^t \left\|e^{(t-\kappa(s))A}B(Z_s)\right\|^2_{L^p(\mathbb{P};\mathrm{HS}(U,H_\gamma))} ds\right]^{1/2}.\end{aligned} \tag{19}$$

Note that, e.g., [16, Theorem 2.5.34 and Lemma 2.5.35] proves that

$$\begin{aligned}&\int_0^t \left\|e^{(t-\kappa(s))A}F(Z_s)\right\|_{L^p(\mathbb{P};H_\gamma)} ds \leq \int_0^t \left\|e^{(t-\kappa(s))A}\right\|_{L(H_{-\alpha},H_\gamma)} \|F(Z_s)\|_{L^p(\mathbb{P};H_{-\alpha})} ds \\ &\leq \int_0^t (t-\kappa(s))^{-(\gamma+\alpha)}\|F(Z_s)\|_{L^p(\mathbb{P};H_{-\alpha})}\, ds \leq C\int_0^t (t-s)^{-(\gamma+\alpha)}\left\|1+\|Z_s\|_H^a\right\|_{L^p(\mathbb{P};\mathbb{R})} ds \\ &\leq C\int_0^t (t-s)^{-(\gamma+\alpha)}\left(1+\|Z_s\|^a_{L^{pa}(\mathbb{P};H)}\right) ds \leq \tfrac{C}{1-(\gamma+\alpha)} t^{1-(\gamma+\alpha)}\left(1+\sup_{s\in[0,t]}\|Z_s\|^a_{L^{pa}(\mathbb{P};H)}\right)\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\int_0^t \left\|e^{(t-\kappa(s))A}B(Z_s)\right\|^2_{L^p(\mathbb{P};\mathrm{HS}(U,H_\gamma))} ds \leq \int_0^t \left\|e^{(t-\kappa(s))A}\right\|^2_{L(H_{-\beta},H_\gamma)} \|B(Z_s)\|^2_{L^p(\mathbb{P};\mathrm{HS}(U,H_{-\beta}))} ds \\ &\leq \int_0^t \tfrac{C^2}{(t-\kappa(s))^{2(\gamma+\beta)}}\left\|1+\|Z_s\|_H^a\right\|^2_{L^p(\mathbb{P};\mathbb{R})} ds \leq \tfrac{C^2}{1-2(\gamma+\beta)} t^{1-2(\gamma+\beta)}\left(1+\sup_{s\in[0,t]}\|Z_s\|^a_{L^{pa}(\mathbb{P};H)}\right)^2.\end{aligned} \tag{21}$$

Putting (20) and (21) into (19) yields (18). The proof of Lemma 3.1 is now completed. □

3.3 A second bootstrap-type argument for a priori bounds

lemma 3.2. *Assume the setting in Section 3.1, let* $\eta \in [0, 1/2)$, $t \in (0,T]$, *and assume that* $\sup_{s\in[0,T]} \big\|\|F(Z_s)\|_H$ $\|B(Z_s)\|_{\mathrm{HS}(U,H)}\big\|_{L^p(\mathbb{P};\mathbb{R})} < \infty$. *Then it holds that* $\mathbb{P}(Y_t \in \cap_{r\in(-\infty,1/2)} H_r) = 1$ *and*

$$\begin{aligned}\|Y_t\|_{L^p(\mathbb{P};H_\eta)} \leq \|e^{tA}\xi\|_{L^p(\mathbb{P};H_\eta)} + \tfrac{1}{1-\eta} t^{1-\eta} \sup_{s\in[0,t]}\|F(Z_s)\|_{L^p(\mathbb{P};H)} \\ + \sqrt{\tfrac{p(p-1)}{2(1-2\eta)}}\, t^{1/2-\eta}\sup_{s\in[0,t]}\|B(Z_s)\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))}.\end{aligned} \tag{22}$$

Proof of Lemma 3.2. First observe that, e.g., Theorem 2.5.34 in [16] proves that for all $r \in [0, 1/2)$ it holds that

$$\begin{aligned}&\int_0^t \|e^{(t-s)A}F(Z_s)\|_{L^p(\mathbb{P};H_r)} ds \leq \sup_{s\in[0,T]}\|F(Z_s)\|_H \int_0^t \|(-A)^r e^{(t-s)A}\|_{L(H)}\, ds \\ &\leq \sup_{s\in[0,T]}\|F(Z_s)\|_H \int_0^t (t-s)^{-r}\, ds = \sup_{s\in[0,T]}\|F(Z_s)\|_H \tfrac{t^{1-r}}{1-r} < \infty\end{aligned} \tag{23}$$

and

$$\begin{aligned}&\sqrt{\int_0^t \|e^{(t-s)A}B(Z_s)\|^2_{\mathrm{HS}(U,H_r)} ds} \leq \sup_{s\in[0,T]}\|B(Z_s)\|_{\mathrm{HS}(U,H)}\sqrt{\int_0^t \|(-A)^r e^{(t-s)A}\|^2_{L(H)} ds} \\ &\leq \sup_{s\in[0,T]}\|B(Z_s)\|_{\mathrm{HS}(U,H)}\sqrt{\int_0^t (t-s)^{-2r} ds} = \sup_{s\in[0,T]}\|B(Z_s)\|_{\mathrm{HS}(U,H)} \tfrac{t^{1/2-r}}{\sqrt{1-2r}} < \infty.\end{aligned} \tag{24}$$

Next note that (23) and (24) prove that $\mathbb{P}(Y_t \in \cap_{r\in(-\infty,1/2)} H_r) = 1$. Moreover, observe that (23), (24), and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [5] imply (22). The proof of Lemma 3.2 is thus completed. □

4 Strong temporal error estimates for nonlinearities-stopped schemes

In this section we estimate temporal discretization errors of nonlinearities-stopped exponential Euler approximations; see Corollary 4.4 below. For this we introduce similar as in Jentzen & Kurniawan [18, (11), (70), (136)] suitable semilinear integrated counterparts of the nonlinearities-stopped exponential Euler approximations. Then we estimate the differences of the nonlinearities-stopped exponential Euler approximations and their semilinear counterparts in a straightforward way (see Lemma 4.2) and we employ the perturbation estimate in Theorem 2.10 in Hutzenthaler & Jentzen [11] to estimate the differences of the solution process of the considered SPDE and the semilinear integrated counterparts of the nonlinearities-stopped exponential Euler approximations (see Lemma 4.3). Combining Lemma 4.2 and Lemma 4.3 with the triangle inequality will then immediately result in Corollary 4.4.

4.1 Setting

Assume the setting in Section 1.2, let $N \in \mathbb{N}$, $F \in \mathcal{M}\big(\mathcal{B}(H_\gamma), \mathcal{B}(H)\big)$, $B \in \mathcal{M}\big(\mathcal{B}(H_\gamma), \mathcal{B}(\mathrm{HS}(U,H))\big)$, assume that $\gamma < \min\{1-\alpha, 1/2 - \beta\}$, let $X\colon [0,T]\times\Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s)\,ds + \int_0^t e^{(t-s)A} B(X_s)\,dW_s, \tag{25}$$

let $Y\colon [0,T]\times\Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process such that for all $t\in[0,T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} Y_t = e^{tA}\xi &+ \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\, \mathbb{1}_{\left\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)} \le \left(\frac{N}{T}\right)^\theta\right\}}\, F(Y_{\lfloor s\rfloor_{T/N}})\,ds \\ &+ \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\, \mathbb{1}_{\left\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)} \le \left(\frac{N}{T}\right)^\theta\right\}}\, B(Y_{\lfloor s\rfloor_{T/N}})\,dW_s, \end{aligned} \tag{26}$$

and let $\bar{Y}\colon [0,T]\times\Omega\to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths such that for all $t\in[0,T]$ it holds $\mathbb{P}$-a.s. that

$$\bar{Y}_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(Y_{\lfloor s\rfloor_{T/N}})\,ds + \int_0^t e^{(t-s)A} B(Y_{\lfloor s\rfloor_{T/N}})\,dW_s. \tag{27}$$

4.2 Analysis of the differences between nonlinearities-stopped exponential Euler approximations and their semilinear counterparts

lemma 4.1. *Assume the setting in Section 1.2 and let $Z \in \mathcal{M}\big(\mathcal{F}, \mathcal{B}(H_\gamma)\big)$, $\kappa \in [\theta/p, \infty)$. Then*

$$\left\|1 - \mathbb{1}_{\{\|F(Z)\|_H + \|B(Z)\|_{HS(U,H)} \le \left(\frac{N}{T}\right)^\theta\}}\right\|_{L^p(\mathbb{P};\mathbb{R})} \le \left(\tfrac{T}{N}\right)^\kappa \Big(\|F(Z)\|_{L^{\kappa p/\theta}(\mathbb{P};H)} + \|B(Z)\|_{L^{\kappa p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big)^{\frac{\kappa}{\theta}}. \tag{28}$$

Proof of Lemma 4.1. Markov's inequality shows that

$$\begin{aligned}
&\big\|1 - \mathbb{1}_{\left\{\|F(Z)\|_H + \|B(Z)\|_{HS(U,H)} \leq \left(\frac{N}{T}\right)^{\theta}\right\}}\big\|_{L^p(\mathbb{P};H)} = \Big|\mathbb{P}\Big[\big(\|F(Z)\|_H + \|B(Z)\|_{HS(U,H)}\big)^{\kappa p/\theta} > \left(\tfrac{N}{T}\right)^{\kappa p}\Big]\Big|^{\frac{1}{p}} \\
&\leq \left(\tfrac{T}{N}\right)^{\kappa} \Big|\mathbb{E}\Big[\big(\,\|F(Z)\|_H + \|B(Z)\|_{HS(U,H)}\,\big)^{\frac{\kappa p}{\theta}}\Big]\Big|^{\frac{1}{p}} = \left(\tfrac{T}{N}\right)^{\kappa} \big\|\|F(Z)\|_H + \|B(Z)\|_{HS(U,H)}\big\|^{\frac{\kappa}{\theta}}_{L^{\kappa p/\theta}(\mathbb{P};\mathbb{R})} \\
&\leq \left(\tfrac{T}{N}\right)^{\kappa} \big(\|F(Z)\|_{L^{\kappa p/\theta}(\mathbb{P};H)} + \|B(Z)\|_{L^{\kappa p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\big)^{\frac{\kappa}{\theta}}.
\end{aligned} \tag{29}$$

The proof of Lemma 4.1 is now completed. ☐

lemma 4.2. *Assume the setting in Section 4.1 and let $\rho \in [\delta, 1-\delta)$, $t \in [0,T]$. Then*

$$\begin{aligned}
&\|Y_t - \bar{Y}_t\|_{L^p(\mathbb{P};H_\delta)} \\
&\leq \tfrac{\max\{1,T^{3/2}\}}{1-\delta-\rho} N^{(\delta-\rho)/2}\Big[1 + \sup_{s\in[0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)} + \tfrac{\sqrt{p(p-1)}}{\sqrt{2}} \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big]^{1+\frac{1}{2\theta}}.
\end{aligned} \tag{30}$$

Proof of Lemma 4.2. Note that

$$\begin{aligned}
&\|Y_t - \bar{Y}_t\|_{L^p(\mathbb{P};H_\delta)} \leq \int_0^t \Big\|\big(e^{(t-\lfloor s\rfloor_{T/N})A} - e^{(t-s)A}\big)F(Y_{\lfloor s\rfloor_{T/N}})\Big\|_{L^p(\mathbb{P};H_\delta)} ds \\
&+ \int_0^t \Big\|e^{(t-s)A}\big(1 - \mathbb{1}_{\left\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^{\theta}\right\}}\big)F(Y_{\lfloor s\rfloor_{T/N}})\Big\|_{L^p(\mathbb{P};H_\delta)} ds \\
&+ \Big\|\int_0^t \big(e^{(t-\lfloor s\rfloor_{T/N})A} - e^{(t-s)A}\big)\,\mathbb{1}_{\left\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^{\theta}\right\}} B(Y_{\lfloor s\rfloor_{T/N}})\, dW_s\Big\|_{L^p(\mathbb{P};H_\delta)} \\
&+ \Big\|\int_0^t e^{(t-s)A}\big(1 - \mathbb{1}_{\left\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^{\theta}\right\}}\big)B(Y_{\lfloor s\rfloor_{T/N}})\, dW_s\Big\|_{L^p(\mathbb{P};H_\delta)}.
\end{aligned} \tag{31}$$

Moreover, observe that, e.g., [16, Theorem 2.5.34 and Lemma 2.5.35] implies that

$$\begin{aligned}
&\int_0^t \Big\|\big(e^{(t-\lfloor s\rfloor_{T/N})A} - e^{(t-s)A}\big)F(Y_{\lfloor s\rfloor_{T/N}})\Big\|_{L^p(\mathbb{P};H_\delta)} ds \\
&\leq \int_0^t \big\|(-A)^{\rho+\delta/2} e^{(t-s)A}\big\|_{L(H)} \big\|(-A)^{\delta/2-\rho}\big(\mathrm{Id}_H - e^{(s-\lfloor s\rfloor_{T/N})A}\big)\big\|_{L(H)} \|F(Y_{\lfloor s\rfloor_{T/N}})\|_{L^p(\mathbb{P};H)}\, ds \\
&\leq \int_0^t \tfrac{(s-\lfloor s\rfloor_{T/N})^{\rho-\delta/2}}{(t-s)^{\rho+\delta/2}} \|F(Y_{\lfloor s\rfloor_{T/N}})\|_{L^p(\mathbb{P};H)}\, ds \leq \tfrac{T^{1-\delta}}{1-\rho-\delta/2} N^{-\rho+\delta/2} \sup_{s\in[0,T]} \|F(Y_s)\|_{L^p(\mathbb{P};H)}.
\end{aligned} \tag{32}$$

In addition, note that Hölder's inequality, e.g., Theorem 2.5.34 in [16], and Lemma 4.1 prove that

$$\begin{aligned}
&\int_0^t \Big\| e^{(t-s)A} \big(1 - \mathbb{1}_{\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \le (\frac{N}{T})^\theta\}}\big) F(Y_{\lfloor s\rfloor_{T/N}}) \Big\|_{L^p(\mathbb{P};H_\delta)} ds \\
&\le \int_0^t \big\| e^{(t-s)A} F(Y_{\lfloor s\rfloor_{T/N}}) \big\|_{L^{2p}(\mathbb{P};H_\delta)} \Big\| 1 - \mathbb{1}_{\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \le (\frac{N}{T})^\theta\}} \Big\|_{L^{2p}(\mathbb{P};\mathbb{R})} ds \\
&\le \int_0^t (t-s)^{-\delta} \big\| F(Y_{\lfloor s\rfloor_{T/N}}) \big\|_{L^{2p}(\mathbb{P};H)} \Big\| 1 - \mathbb{1}_{\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \le (\frac{N}{T})^\theta\}} \Big\|_{L^{2p}(\mathbb{P};\mathbb{R})} ds \qquad (33) \\
&\le \tfrac{t^{1-\delta}}{1-\delta} \sqrt{\tfrac{T}{N}} \sup_{s\in[0,T]} \|F(Y_s)\|_{L^{2p}(\mathbb{P};H)} \left[\sup_{s\in[0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)} + \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{\frac{1}{2\theta}} \\
&\le \tfrac{T^{3/2-\delta}}{1-\delta} N^{-1/2} \sup_{s\in[0,T]} \|F(Y_s)\|_{L^{2p}(\mathbb{P};H)} \left[\sup_{s\in[0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)} + \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{\frac{1}{2\theta}}.
\end{aligned}$$

Furthermore, the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [5], and, e.g., [16, Theorem 2.5.34 and Lemma 2.5.35] show that

$$\begin{aligned}
&\Big\| \int_0^t \big(e^{(t-\lfloor s\rfloor_{T/N})A} - e^{(t-s)A}\big) \mathbb{1}_{\{\|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \le (\frac{N}{T})^\theta\}} B(Y_{\lfloor s\rfloor_{T/N}})\, dW_s \Big\|_{L^p(\mathbb{P};H_\delta)} \\
&\le \sqrt{\tfrac{p(p-1)}{2} \int_0^t \big\| \big(e^{(t-\lfloor s\rfloor_{T/N})A} - e^{(t-s)A}\big) B(Y_{\lfloor s\rfloor_{T/N}}) \big\|^2_{L^p(\mathbb{P};\mathrm{HS}(U,H_\delta))} ds} \\
&\le \sqrt{\tfrac{p(p-1)}{2} \int_0^t \big\| (-A)^{\frac{\rho+\delta}{2}} e^{(t-s)A} \big\|^2_{L(H)} \big\| (-A)^{\frac{-\rho+\delta}{2}} \big(\mathrm{Id}_H - e^{(s-\lfloor s\rfloor_{T/N})A}\big) \big\|^2_{L(H)} \|B(Y_{\lfloor s\rfloor_{T/N}})\|^2_{L^p(\mathbb{P};\mathrm{HS}(U,H))} ds} \\
&\le \tfrac{\sqrt{p(p-1)}}{\sqrt{2}} \sup\nolimits_{s\in[0,T]} \|B(Y_s)\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))} \sqrt{\int_0^t \tfrac{(s-\lfloor s\rfloor_{T/N})^{\rho-\delta}}{(t-s)^{\rho+\delta}} ds} \qquad (34) \\
&\le \sqrt{\tfrac{p(p-1)}{2(1-\rho-\delta)}} T^{1/2-\delta} N^{\frac{-\rho+\delta}{2}} \sup\nolimits_{s\in[0,T]} \|B(Y_s)\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))}.
\end{aligned}$$

Moreover, the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [5],

Hölder's inequality, Lemma 4.1, and, e.g., Theorem 2.5.34 in [16] prove that

$$
\begin{aligned}
&\tfrac{2}{p(p-1)} \left\| \int_0^t e^{(t-s)A} \big(1 - \mathbb{1}_{\left\{\|F(Y_{\lfloor s \rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s \rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} \big) B(Y_{\lfloor s \rfloor_{T/N}}) \, dW_s \right\|^2_{L^p(\mathbb{P};H_\delta)} \\
&\leq \int_0^t \left\| e^{(t-s)A} \big(1 - \mathbb{1}_{\left\{\|F(Y_{\lfloor s \rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s \rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} \big) B(Y_{\lfloor s \rfloor_{T/N}}) \right\|^2_{L^p(\mathbb{P};\mathrm{HS}(U,H_\delta))} ds \\
&\leq \int_0^t \big\| e^{(t-s)A} B(Y_{\lfloor s \rfloor_{T/N}}) \big\|^2_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H_\delta))} \big\| 1 - \mathbb{1}_{\left\{\|F(Y_{\lfloor s \rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s \rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} \big\|^2_{L^{2p}(\mathbb{P};\mathbb{R})} ds \\
&\leq \int_0^t (t-s)^{-2\delta} \|B(Y_{\lfloor s \rfloor_{T/N}})\|^2_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} \big\| 1 - \mathbb{1}_{\left\{\|F(Y_{\lfloor s \rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s \rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} \big\|^2_{L^{2p}(\mathbb{P};\mathbb{R})} ds \\
&\leq \tfrac{T t^{1-2\delta}}{(1-2\delta)N} \sup_{s\in[0,T]} \|B(Y_s)\|^2_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} \left[\sup_{s\in[0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)} + \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{\frac{1}{\theta}} \\
&\leq \tfrac{T^{2(1-\delta)}}{N(1-2\delta)} \sup_{s\in[0,T]} \|B(Y_s)\|^2_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} \left[\sup_{s\in[0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)} + \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{\frac{1}{\theta}} .
\end{aligned}
\tag{35}
$$

Combining (31)–(35) shows that

$$
\begin{aligned}
&\|Y_t - \bar{Y}_t\|_{L^p(\mathbb{P};H_\delta)} \leq \tfrac{1}{\sqrt{N}} \left[\sup_{s\in[0,T]} \|F(Y_s)\|_{L^{p/\theta}(\mathbb{P};H)} + \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{p/\theta}(\mathbb{P};\mathrm{HS}(U,H))} \right]^{\frac{1}{2\theta}} \\
&\cdot \left[\tfrac{T^{3/2-\delta}}{1-\delta} \sup_{s\in[0,T]} \|F(Y_s)\|_{L^{2p}(\mathbb{P};H)} + \sqrt{\tfrac{p(p-1)}{2(1-2\delta)}} T^{1-\delta} \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} \right] \\
&+ \tfrac{T^{1-\delta}}{1-\rho-\frac{\delta}{2}} N^{\frac{\delta-2\rho}{2}} \sup_{s\in[0,T]} \|F(Y_s)\|_{L^p(\mathbb{P};H)} + \sqrt{\tfrac{p(p-1)}{2(1-\rho-\delta)}} T^{\frac{1}{2}-\delta} N^{\frac{\delta-\rho}{2}} \sup_{s\in[0,T]} \|B(Y_s)\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))}.
\end{aligned}
\tag{36}
$$

This completes the proof of Lemma 4.2. □

4.3 Analysis of the differences between semilinear integrated nonlinearities-stopped exponential Euler approximations and solution processes of stochastic evolution equations

lemma 4.3. *Assume the setting in Section 4.1, let* $\rho \in [\delta, 1-\delta)$, $\eta \in [\delta, \frac{1}{2})$, $\varepsilon \in (0,\infty)$, *assume that* $\sup_{h\in\mathbb{H}} |\lambda_h| < \infty$, *and assume that for all* $x, y \in H_\gamma$ *it holds that* $\max\{\|F(x) - F(y)\|_H, \|B(x) - B(y)\|_{\mathrm{HS}(U,H)}\} \leq C\|x-y\|_{H_\delta}(1+\|x\|^c_{H_\gamma}+\|y\|^c_{H_\gamma})$ *and* $\langle x-y, Ax+F(x)-Ay-F(y)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|B(x)-B(y)\|^2_{\mathrm{HS}(U,H)} \leq C\|x-y\|^2_H$. *Then*

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \bar{Y}_t\|_{L^p(\mathbb{P};H)} \leq N^{\delta - \min\{\eta, \frac{\rho+\delta}{2}\}} \tfrac{\max\{1,T^2\}}{(1-\delta-\rho)} \big(C^2(1+1/\varepsilon)p\big)^{1/p} \exp\!\Big(\tfrac{TC^2 p(1+1/\varepsilon)}{2}\Big) \\
&\cdot \Bigg(2\left[1 + \sup_{s\in[0,T]} \|F(Y_s)\|_{L^{2p/\theta}(\mathbb{P};H)} + \sqrt{p(2p-1)} \sup_{s\in[0,T]} \|B(Y_s)\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\right]^{1+\frac{1}{2\theta}} \\
&\quad + \sup_{s\in[0,T]} \|Y_s\|_{L^{2p}(\mathbb{P};H_\eta)} \Bigg) \Big(1 + \sup_{s\in[0,T]} \|\bar{Y}_s\|^c_{L^{2pc}(\mathbb{P};H_\gamma)} + \sup_{s\in[0,T]} \|Y_s\|^c_{L^{2pc}(\mathbb{P};H_\gamma)}\Big).
\end{aligned}
\tag{37}
$$

Proof of Lemma 4.3. Throughout this proof let $\chi \in [0,\infty)$ be the real number given by $\chi = \frac{C(p-1)}{p}(1 + \frac{C(p-2)(1+1/\varepsilon)}{2})$. We intend to prove Lemma 4.3 through an application of Theorem 2.10

in Hutzenthaler & Jentzen [11]. To this end we now check the assumptions of Theorem 2.10 in Hutzenthaler & Jentzen [11]. Let $\tilde{X}\colon [0,T]\times\Omega\to H_\gamma$ be the stochastic process which satisfies that for all $s\in[0,T]$ it holds that $\tilde{X}_s = e^{-sA}X_s$. Itô's formula then proves that for all $s\in[0,T]$ it holds $\mathbb{P}$-a.s. that $X_s = e^{sA}\tilde{X}_s = \xi + \int_0^s [AX_u + F(X_u)]\,du + \int_0^s B(X_u)\,dW_u$. Similar we see that for all $s\in[0,T]$ it holds $\mathbb{P}$-a.s. that $\bar{Y}_s = \xi + \int_0^s [A\bar{Y}_u + F(Y_{\lfloor u\rfloor_{T/N}})]\,du + \int_0^s B(Y_{\lfloor u\rfloor_{T/N}})\,dW_u$. Next observe that for all $x\in H_\gamma$ it holds that $\|F(x)\|_H \le \|F(0)\|_H + C\|x\|_{H_\delta}(1+\|x\|_{H_\gamma}^c)$ and $\|B(x)\|_{\mathrm{HS}(U,H)} \le \|B(0)\|_{\mathrm{HS}(U,H)} + C\|x\|_{H_\delta}(1+\|x\|_{\mathrm{HS}(U,H_\gamma)}^c)$. Combining this with the continuity of X and $\bar{Y}$ implies that $\int_0^T \|A\bar{Y}_s\|_H + \|F(Y_{\lfloor s\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor s\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)}^2 + \|F(\bar{Y}_s)\|_H + \|B(\bar{Y}_s)\|_{\mathrm{HS}(U,H)}^2 + \|AX_s\|_H + \|F(X_s)\|_H + \|B(X_s)\|_{\mathrm{HS}(U,H)}^2\,ds < \infty$. We can thus apply Theorem 2.10 in Hutzenthaler & Jentzen [11] to obtain that

$$\begin{aligned}\sup\nolimits_{t\in[0,T]}\|X_t-\bar{Y}_t\|_{L^p(\mathbb{P};H)} \le e^{(C+\chi)T}\Big\|&p\|X-\bar{Y}\|_H^{p-2}\big[\langle X-\bar{Y}, F(\bar{Y})-F(Y_{\lfloor\cdot\rfloor_{T/N}})\rangle_H\\ &+\tfrac{(p-1)(1+1/\varepsilon)}{2}\|B(Y_{\lfloor\cdot\rfloor_{T/N}})-B(\bar{Y})\|_{\mathrm{HS}(U,H)}^2 - \chi\|X-\bar{Y}\|_H^2\big]^+\Big\|_{L^1(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})}^{1/p}.\end{aligned} \tag{38}$$

The Cauchy-Schwarz inequality therefore implies that

$$\begin{aligned}\sup\nolimits_{t\in[0,T]}\|X_t-\bar{Y}_t\|_{L^p(\mathbb{P};H)} \le e^{(C+\chi)T}\Big\|&p\|X-\bar{Y}\|_H^{p-2}\big[\|X-\bar{Y}\|_H\|F(\bar{Y})-F(Y_{\lfloor\cdot\rfloor_{T/N}})\|_H\\ &+\tfrac{(p-1)(1+1/\varepsilon)}{2}\|B(Y_{\lfloor\cdot\rfloor_{T/N}})-B(\bar{Y})\|_{\mathrm{HS}(U,H)}^2 - \chi\|X-\bar{Y}\|_H^2\big]^+\Big\|_{L^1(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})}^{1/p}.\end{aligned} \tag{39}$$

Next note that the assumption that $\forall\, x,y\in H_\gamma\colon \|F(x)-F(y)\|_H \le C\|x-y\|_{H_\delta}(1+\|x\|_{H_\gamma}^c+\|y\|_{H_\gamma}^c)$ and Young's inequality imply that for all $s\in[0,T]$ it holds that

$$\begin{aligned}&\|X_s-\bar{Y}_s\|_H^{p-1}\|F(\bar{Y}_s)-F(Y_{\lfloor s\rfloor_{T/N}})\|_H\\ &\le \tfrac{C(p-1)}{p}\|X_s-\bar{Y}_s\|_H^p + \tfrac{C}{p}\|\bar{Y}_s-Y_{\lfloor s\rfloor_{T/N}}\|_{H_\delta}^p\big(1+\|\bar{Y}_s\|_{H_\gamma}^c+\|Y_{\lfloor s\rfloor_{T/N}}\|_{H_\gamma}^c\big)^p.\end{aligned} \tag{40}$$

Moreover, note that the assumption that $\forall\, x,y\in H_\gamma\colon \|B(x)-B(y)\|_{\mathrm{HS}(U,H)} \le C\|x-y\|_{H_\delta}(1+\|x\|_{H_\gamma}^c+\|y\|_{H_\gamma}^c)$ and again Young's inequality imply that for all $s\in[0,T]$ it holds that

$$\begin{aligned}&\|X_s-\bar{Y}_s\|_H^{p-2}\|B(Y_{\lfloor s\rfloor_{T/N}})-B(\bar{Y}_s)\|_{\mathrm{HS}(U,H)}^2\\ &\le \tfrac{C^2(p-2)}{p}\|X_s-\bar{Y}_s\|_H^p + \tfrac{2C^2}{p}\|Y_{\lfloor s\rfloor_{T/N}}-\bar{Y}_s\|_{H_\delta}^p\big(1+\|\bar{Y}_s\|_{H_\gamma}^c+\|Y_{\lfloor s\rfloor_{T/N}}\|_{H_\gamma}^c\big)^p.\end{aligned} \tag{41}$$

Combining (39)–(41) with Hölder's inequality shows that

$$\begin{aligned}&\sup\nolimits_{t\in[0,T]}\|X_t-\bar{Y}_t\|_{L^p(\mathbb{P};H)} \le e^{(C+\chi)T}\Big\|C\|\bar{Y}-Y_{\lfloor\cdot\rfloor_{T/N}}\|_{H_\delta}^p\big(1+\|\bar{Y}\|_{H_\gamma}^c+\|Y_{\lfloor\cdot\rfloor_{T/N}}\|_{H_\gamma}^c\big)^p\\ &\quad+C^2(p-1)(1+1/\varepsilon)\|Y_{\lfloor\cdot\rfloor_{T/N}}-\bar{Y}\|_{H_\delta}^p\big(1+\|\bar{Y}\|_{H_\gamma}^{\bar{c}}+\|Y_{\lfloor\cdot\rfloor_{T/N}}\|_{H_\gamma}^c\big)^p\Big\|_{L^1(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})}^{\frac{1}{p}}\\ &= e^{(C+\chi)T}\big(C+C^2(p-1)(1+1/\varepsilon)\big)^{1/p}\Big\|\|\bar{Y}-Y_{\lfloor\cdot\rfloor_{T/N}}\|_{H_\delta}\big(1+\|\bar{Y}\|_{H_\gamma}^c+\|Y_{\lfloor\cdot\rfloor_{T/N}}\|_{H_\gamma}^c\big)\Big\|_{L^p(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})}\\ &\le e^{(C+\chi)T}\big(TC^2p(1+1/\varepsilon)\big)^{1/p}\sup\nolimits_{s\in[0,t]}\big\|\bar{Y}_s-Y_{\lfloor s\rfloor_{T/N}}\big\|_{L^{2p}(\mathbb{P};H_\delta)}\\ &\qquad\cdot\big(1+\sup\nolimits_{s\in[0,T]}\|\bar{Y}_s\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c+\sup\nolimits_{s\in[0,T]}\|Y_{\lfloor s\rfloor_{T/N}}\|_{L^{2pc}(\mathbb{P};H_\gamma)}^c\big).\end{aligned} \tag{42}$$

Next observe that the triangle inequality implies that

$$\sup_{s\in[0,T]} \|\bar{Y}_s - Y_{\lfloor s\rfloor_{T/N}}\|_{L^{2p}(\mathbb{P};H_\delta)} \leq \sup_{s\in[0,T]} \|\bar{Y}_s - Y_s\|_{L^{2p}(\mathbb{P};H_\delta)} + \sup_{s\in[0,T]} \|Y_s - Y_{\lfloor s\rfloor_{T/N}}\|_{L^{2p}(\mathbb{P};H_\delta)}. \tag{43}$$

In addition, observe that the triangle inequality proves that for all $s \in [0,T]$ it holds that

$$\begin{aligned}
&\|Y_s - Y_{\lfloor s\rfloor_{T/N}}\|_{L^{2p}(\mathbb{P};H_\delta)} \leq \left\|(e^{(s-\lfloor s\rfloor_{T/N})A} - \mathrm{Id}_H) Y_{\lfloor s\rfloor_{T/N}}\right\|_{L^{2p}(\mathbb{P};H_\delta)} \\
&+ \left\| \int_{\lfloor s\rfloor_{T/N}}^s e^{(s-\lfloor u\rfloor_{T/N})A} \mathbb{1}_{\left\{\|F(Y_{\lfloor u\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor u\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} F(Y_{\lfloor u\rfloor_{T/N}})\, du \right\|_{L^{2p}(\mathbb{P};H_\delta)} \\
&+ \left\| \int_{\lfloor s\rfloor_{T/N}}^s e^{(s-\lfloor u\rfloor_{T/N})A} \mathbb{1}_{\left\{\|F(Y_{\lfloor u\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor u\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} B(Y_{\lfloor u\rfloor_{T/N}})\, dW_u \right\|_{L^{2p}(\mathbb{P};H_\delta)}.
\end{aligned} \tag{44}$$

Furthermore, observe that, e.g., [16, Lemma 2.5.35] proves that for all $s \in [0,T]$ it holds that

$$\begin{aligned}
&\left\|(e^{(s-\lfloor s\rfloor_{T/N})A} - \mathrm{Id}_H) Y_{\lfloor s\rfloor_{T/N}}\right\|_{L^{2p}(\mathbb{P};H_\delta)} \leq \sup\nolimits_{u\in[0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)} \left\|e^{(s-\lfloor s\rfloor_{T/N})A} - \mathrm{Id}_H\right\|_{L(H_\eta,H_\delta)} \\
&\leq \sup\nolimits_{u\in[0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)} \left(s - \lfloor s\rfloor_{T/N}\right)^{\eta-\delta} \leq \sup\nolimits_{u\in[0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)} \left(\tfrac{T}{N}\right)^{\eta-\delta}.
\end{aligned} \tag{45}$$

Moreover, note that, e.g., Theorem 2.5.34 in [16] proves that for all $s \in [0,T]$ it holds that

$$\begin{aligned}
&\left\| \int_{\lfloor s\rfloor_{T/N}}^s e^{(s-\lfloor u\rfloor_{T/N})A} \mathbb{1}_{\left\{\|F(Y_{\lfloor u\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor u\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} F(Y_{\lfloor u\rfloor_{T/N}})\, du \right\|_{L^{2p}(\mathbb{P};H_\delta)} \\
&\leq \int_{\lfloor s\rfloor_{T/N}}^s \left\| e^{(s-\lfloor u\rfloor_{T/N})A} F(Y_{\lfloor u\rfloor_{T/N}}) \right\|_{L^{2p}(\mathbb{P};H_\delta)} du \\
&\leq \sup\nolimits_{u\in[0,T]} \|F(Y_u)\|_{L^{2p}(\mathbb{P};H)} \int_{\lfloor s\rfloor_{T/N}}^s (s - \lfloor u\rfloor_{T/N})^{-\delta}\, du \leq \sup\nolimits_{u\in[0,T]} \|F(Y_u)\|_{L^{2p}(\mathbb{P};H)} \left(\tfrac{T}{N}\right)^{1-\delta}.
\end{aligned} \tag{46}$$

The Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [5] and, e.g., Theorem 2.5.34 in [16] prove that for all $s \in [0,T]$ it holds that

$$\begin{aligned}
&\left\| \int_{\lfloor s\rfloor_{T/N}}^s e^{(s-\lfloor u\rfloor_{T/N})A} \mathbb{1}_{\left\{\|F(Y_{\lfloor u\rfloor_{T/N}})\|_H + \|B(Y_{\lfloor u\rfloor_{T/N}})\|_{\mathrm{HS}(U,H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} B(Y_{\lfloor u\rfloor_{T/N}})\, dW_u \right\|_{L^{2p}(\mathbb{P};H_\delta)} \\
&\leq \sqrt{p(2p-1) \int_{\lfloor s\rfloor_{T/N}}^s \left\| e^{(s-\lfloor u\rfloor_{T/N})A} B(Y_{\lfloor u\rfloor_{T/N}}) \right\|^2_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H_\delta))} du} \\
&\leq \sqrt{p(2p-1)}\, \sup\nolimits_{u\in[0,T]} \|B(Y_u)\|_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} \left(\tfrac{T}{N}\right)^{1/2-\delta}.
\end{aligned} \tag{47}$$

Combining (44)–(47) implies that

$$\begin{aligned}
&\sup\nolimits_{t\in[0,T]} \|Y_t - Y_{\lfloor t\rfloor_{T/N}}\|_{L^{2p}(\mathbb{P};H_\delta)} \leq \left(\tfrac{T}{N}\right)^{1/2-\delta} \sqrt{p(2p-1)}\, \sup\nolimits_{u\in[0,T]} \|B(Y_u)\|_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} \\
&+ \left(\tfrac{T}{N}\right)^{\eta-\delta} \sup\nolimits_{u\in[0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)} + \left(\tfrac{T}{N}\right)^{1-\delta} \sup\nolimits_{u\in[0,T]} \|F(Y_u)\|_{L^{2p}(\mathbb{P};H)}.
\end{aligned} \tag{48}$$

Furthermore, note that Lemma 4.2 proves that

$$\begin{aligned}
&\sup\nolimits_{t\in[0,T]} \|\bar{Y}_t - Y_t\|_{L^{2p}(\mathbb{P};H_\delta)} \\
&\leq \tfrac{\max\{1,T^{3/2}\}}{1-\delta-\rho} N^{(\delta-\rho)/2} \Big[1 + \sup_{u\in[0,T]} \|F(Y_u)\|_{L^{2p/\theta}(\mathbb{P};H)} + \sqrt{p(2p-1)} \sup_{u\in[0,T]} \|B(Y_u)\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big]^{1+\frac{1}{2\theta}}.
\end{aligned} \tag{49}$$

Combining (43), (48), and (49) shows that

$$\begin{aligned}&\sup_{t\in[0,T]} \|\bar{Y}_t - Y_{\lfloor t\rfloor_{T/N}}\|_{L^{2p}(\mathbb{P};H_\delta)} \le \left(\tfrac{T}{N}\right)^{\eta-\delta} \sup_{u\in[0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)}\\ &+ \left(\tfrac{T}{N}\right)^{1-\delta} \sup_{u\in[0,T]} \|F(Y_u)\|_{L^{2p}(\mathbb{P};H)} + \left(\tfrac{T}{N}\right)^{1/2-\delta} \sqrt{p(2p-1)} \sup_{u\in[0,T]} \|B(Y_u)\|_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))}\\ &+ \tfrac{\max\{1,T^{3/2}\}N^{(\delta-\rho)/2}}{1-\delta-\rho}\Big[1+\sup_{u\in[0,T]} \|F(Y_u)\|_{L^{2p/\theta}(\mathbb{P};H)} + \sqrt{p(2p-1)} \sup_{u\in[0,T]} \|B(Y_u)\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big]^{1+\frac{1}{2\theta}}.\end{aligned} \tag{50}$$

Combining (42) and (50) implies that

$$\begin{aligned}&\sup_{t\in[0,T]} \|X_t - \bar{Y}_t\|_{L^p(\mathbb{P};H)} \le e^{\frac{TC^2p(1+1/\varepsilon)}{2}} \left(TC^2p(1+{}^1\!/\!{}_\varepsilon)\right)^{1/p}\\ &\cdot \Bigg[\left(\tfrac{T}{N}\right)^{1/2-\delta}\sqrt{p(2p-1)} \sup_{u\in[0,T]} \|B(Y_u)\|_{L^{2p}(\mathbb{P};\mathrm{HS}(U,H))} + \left(\tfrac{T}{N}\right)^{\eta-\delta}\sup_{u\in[0,T]} \|Y_u\|_{L^{2p}(\mathbb{P};H_\eta)}\\ &+ \tfrac{\max\{1,T^{3/2}\}N^{(\delta-\rho)/2}}{1-\delta-\rho}\Big[1+\sup_{u\in[0,T]} \|F(Y_u)\|_{L^{2p/\theta}(\mathbb{P};H)} + \sqrt{p(2p-1)} \sup_{u\in[0,T]} \|B(Y_u)\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big]^{1+\frac{1}{2\theta}}\\ &+ \left(\tfrac{T}{N}\right)^{1-\delta} \sup_{u\in[0,T]} \|F(Y_u)\|_{L^{2p}(\mathbb{P};H)}\Bigg] \Big[1+\sup_{u\in[0,T]} \|\bar{Y}_u\|^c_{L^{2pc}(\mathbb{P};H_\gamma)} + \sup_{u\in[0,T]} \|Y_u\|^c_{L^{2pc}(\mathbb{P};H_\gamma)}\Big].\end{aligned} \tag{51}$$

The proof of Lemma 4.3 is thus completed. □

4.4 Analysis of the differences between nonlinearities-stopped exponential Euler approximations and solution processes of stochastic evolution equations

Corollary 4.4. *Assume the setting in Section 4.1, let* $\eta \in [\delta, \frac{1}{2})$, $\varepsilon \in (0,\infty)$, *assume that* $\sup_{h\in\mathbb{H}} |\lambda_h| < \infty$, *and assume that for all* $x, y \in H_\gamma$ *it holds that* $\max\{\|F(x)-F(y)\|_H, \|B(x)-B(y)\|_{\mathrm{HS}(U,H)}\} \le C\|x-y\|_{H_\delta}(1+\|x\|^c_{H_\gamma}+\|y\|^c_{H_\gamma})$ *and* $\langle x-y, Ax+F(x)-Ay-F(y)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|B(x)-B(y)\|^2_{\mathrm{HS}(U,H)} \le C\|x-y\|^2_H$. *Then*

$$\begin{aligned}&\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^p(\mathbb{P};H)}\\ &\le N^{\delta-\eta}\tfrac{\max\{1,T^2\}}{(1-2\eta)}\left(C^2(1+{}^1\!/\!{}_\varepsilon)p\right)^{1/p} \exp\!\Big(\tfrac{TC^2p(1+1/\varepsilon)}{2}\Big) \Bigg(3\Big[1+\sup_{s\in[0,T]} \|F(Y_s)\|_{L^{2p/\theta}(\mathbb{P};H)}\\ &+ \sqrt{p(2p-1)}\sup_{s\in[0,T]} \|B(Y_s)\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big]^{1+\frac{1}{2\theta}} + \sup_{s\in[0,T]} \|Y_s\|_{L^{2p}(\mathbb{P};H_\eta)}\Bigg)\\ &\cdot \left(1+2\left[\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C\Big[\tfrac{T^{1-(\gamma+\alpha)}}{1-(\gamma+\alpha)} + \sqrt{\tfrac{pc(2pc-1)}{(1-2(\gamma+\beta))}}T^{1/2-(\gamma+\beta)}\Big]\Big[1+\sup_{s\in[0,T]} \|Y_s\|^a_{L^{2pca}(\mathbb{P};H)}\Big]\right]^c\right).\end{aligned} \tag{52}$$

Proof of Corollary 4.4. Note that

$$\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^p(\mathbb{P};H)} \le \sup_{t\in[0,T]} \left(\|X_t - \bar{Y}_t\|_{L^p(\mathbb{P};H)} + \|Y_t - \bar{Y}_t\|_{L^p(\mathbb{P};H)}\right). \tag{53}$$

Combining Lemma 4.2, Lemma 4.3 (with $\rho = 2\eta - \delta$ in the notation of Lemma 4.3), and (53) proves that

$$\begin{aligned}&\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^p(\mathbb{P};H)} \le N^{\delta-\eta}\tfrac{\max\{1,T^2\}}{(1-2\eta)}\left(C^2(1+{}^1\!/\!{}_\varepsilon)p\right)^{1/p} \exp\!\Big(\tfrac{TC^2p(1+1/\varepsilon)}{2}\Big)\\ &\cdot \Bigg(3\Big[1+\sup_{s\in[0,T]} \|F(Y_s)\|_{L^{2p/\theta}(\mathbb{P};H)} + \sqrt{p(2p-1)}\sup_{s\in[0,T]} \|B(Y_s)\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big]^{1+\frac{1}{2\theta}}\\ &+ \sup_{s\in[0,T]} \|Y_s\|_{L^{2p}(\mathbb{P};H_\eta)}\Bigg)\Big(1+\sup_{s\in[0,T]} \|\bar{Y}_s\|^c_{L^{2pc}(\mathbb{P};H_\gamma)} + \sup_{s\in[0,T]} \|Y_s\|^c_{L^{2pc}(\mathbb{P};H_\gamma)}\Big).\end{aligned} \tag{54}$$

Moreover, Lemma 3.1 proves that

$$
\begin{aligned}
&\sup_{s\in[0,T]} \|\bar{Y}_s\|^c_{L^{2pc}(\mathbb{P};H_\gamma)} + \sup_{s\in[0,T]} \|Y_s\|^c_{L^{2pc}(\mathbb{P};H_\gamma)} \\
&\leq 2\left[\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C\left[\tfrac{T^{1-(\gamma+\alpha)}}{1-(\gamma+\alpha)} + \sqrt{\tfrac{pc(2pc-1)}{(1-2(\gamma+\beta))}}T^{1/2-(\gamma+\beta)}\right]\left[1+\sup_{s\in[0,T]}\|Y_s\|^a_{L^{2pca}(\mathbb{P};H)}\right]\right]^c.
\end{aligned}
\tag{55}
$$

Combining (54) and (55) completes the proof of Corollary 4.4. □

5 Temporal regularity properties of solution processes of SPDEs

In this section we present a few elementary and essentially well-known temporal regularity properties for solution processes of stochastic partial differential equations with globally Lipschitz continuous coefficients. In the literature similar results can, e.g., be found in Van Neerven et al. [30, Theorem 6.3] and in the references mentioned in Van Neerven et al. [30].

5.1 Setting

Assume the setting in Section 1.2, let $b \in [0,\infty)$, $\eta \in [0,1)$, $F \in \mathcal{C}(H_\gamma, H_{\gamma-\eta})$, $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U, H_{\gamma-\eta/2}))$, assume that for all $x,y \in H_\gamma$ it holds that $\max\{\|F(x)-F(y)\|_{H_{\gamma-\eta}}, \|B(x)-B(y)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}\} \leq b\|x-y\|_{H_\gamma}$, and let $X\colon [0,T]\times\Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic process such that for all $t\in[0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^t \|e^{(t-s)A}F(X_s)\|_H + \|e^{(t-s)A}B(X_s)\|^2_{\mathrm{HS}(U,H)}\,ds < \infty$ and

$$
X_t = \xi + \int_0^t e^{(t-s)A}F(X_s)\,ds + \int_0^t e^{(t-s)A}B(X_s)\,dW_s. \tag{56}
$$

5.2 Temporal regularity properties

lemma 5.1. *Assume the setting in Section 5.1 and let $\beta \in [0, {}^{(1-\eta)}/_{2})$. Then*

$$
\begin{aligned}
&\sup_{s\in[0,T),t\in(s,T]} \frac{\|(X_t - e^{tA}\xi) - (X_s - e^{sA}\xi)\|_{L^p(\mathbb{P};H_\gamma)}}{(t-s)^\beta} \\
&\leq \tfrac{\max\{1,T\}\sqrt{2p(p-1)}}{1-\eta-2\beta}\left(\|F(0)\|_{H_{\gamma-\eta}} + \|B(0)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})} + 2b\sup_{u\in[0,T]}\|X_u\|_{L^p(\mathbb{P};H_\gamma)}\right).
\end{aligned}
\tag{57}
$$

Proof of Lemma 5.1. First observe that, e.g., [16, Theorem 2.5.34 and Lemma 2.5.35] shows that for all $s\in[0,T]$, $t\in[s,T]$ it holds that

$$
\begin{aligned}
&\left\|\int_0^s (e^{(t-u)A} - e^{(s-u)A})F(X_u)\,du\right\|_{L^p(\mathbb{P};H_\gamma)} \leq \int_0^s \left\|(e^{(t-u)A} - e^{(s-u)A})F(X_u)\right\|_{L^p(\mathbb{P};H_\gamma)}\,du \\
&\leq \int_0^s \|(-A)^{\eta+\beta}e^{(s-u)A}\|_{L(H)}\|(-A)^{-\beta}(\mathrm{Id}_H - e^{(t-s)A})\|_{L(H)}\|F(X_u)\|_{L^p(\mathbb{P};H_{\gamma-\eta})}\,du \\
&\leq \int_0^s \tfrac{(t-s)^\beta}{(s-u)^{\eta+\beta}}\left\|\|F(0)\|_{H_{\gamma-\eta}} + b\|X_u\|_{H_\gamma}\right\|_{L^p(\mathbb{P};\mathbb{R})}\,du \\
&\leq \left(\|F(0)\|_{H_{\gamma-\eta}} + b\sup_{u\in[0,T]}\|X_u\|_{L^p(\mathbb{P};H_\gamma)}\right)\tfrac{s^{1-\eta-\beta}}{1-\eta-\beta}(t-s)^\beta.
\end{aligned}
\tag{58}
$$

Moreover, e.g., [16, Theorem 2.5.34 and Lemma 2.5.35] combined with the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [5] proves that for all $s \in [0,T]$, $t \in [s,T]$ it holds that

$$\begin{aligned}
&\left\| \int_0^s (e^{(t-u)A} - e^{(s-u)A}) B(X_u) \, du \right\|_{L^p(\mathbb{P};H_\gamma)} \\
&\leq \sqrt{\tfrac{p(p-1)}{2} \int_0^s \|(e^{(t-u)A} - e^{(s-u)A}) B(X_u)\|^2_{L^p(\mathbb{P};\mathrm{HS}(U,H_\gamma))} \, du} \\
&\leq \sqrt{\tfrac{p(p-1)}{2} \int_0^s \|(-A)^{\eta/2+\beta} e^{(s-u)A}\|^2_{L(H)} \|(-A)^{-\beta}(\mathrm{Id}_H - e^{(t-s)A})\|^2_{L(H)} \|B(X_u)\|^2_{L^p(\mathbb{P};\mathrm{HS}(U,H_{\gamma-\eta/2}))} \, du} \\
&\leq \sqrt{\tfrac{p(p-1)}{2} \int_0^s \tfrac{(t-s)^{2\beta}}{(s-u)^{\eta+2\beta}} \left\| \|B(0)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})} + b\|X_u\|_{H_\gamma} \right\|^2_{L^p(\mathbb{P};\mathbb{R})} \, du} \\
&\leq \left(\|B(0)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})} + b \sup\nolimits_{u\in[0,T]} \|X_u\|_{L^p(\mathbb{P};H_\gamma)} \right) \tfrac{\sqrt{p(p-1)} s^{1/2-\eta/2-\beta}}{\sqrt{2(1-\eta-2\beta)}} (t-s)^\beta.
\end{aligned} \tag{59}$$

Furthermore, note that, e.g., [16, Theorem 2.5.34 and Lemma 2.5.35] implies that for all $s \in [0,T]$, $t \in [s,T]$ it holds that

$$\begin{aligned}
&\left\| \int_s^t e^{(t-u)A} F(X_u) \, du \right\|_{L^p(\mathbb{P};H_\gamma)} \leq \int_s^t \left\| e^{(t-u)A} F(X_u) \right\|_{L^p(\mathbb{P};H_\gamma)} \, du \\
&\leq \int_s^t \|(-A)^\eta e^{(t-u)A}\|_{L(H)} \|F(X_u)\|_{L^p(\mathbb{P};H_{\gamma-\eta})} \, du \leq \int_s^t \tfrac{1}{(t-u)^\eta} (\|F(0)\|_{H_{\gamma-\eta}} + b\|X_u\|_{L^p(\mathbb{P};H_\gamma)}) \, du \\
&\leq \left(\|F(0)\|_{H_{\gamma-\eta}} + b \sup\nolimits_{u\in[0,T]} \|X_u\|_{L^p(\mathbb{P};H_\gamma)} \right) \tfrac{1}{1-\eta} (t-s)^{1-\eta}.
\end{aligned} \tag{60}$$

Again, e.g., [16, Theorem 2.5.34 and Lemma 2.5.35] and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [5] show that for all $s \in [0,T]$, $t \in [s,T]$ it holds that

$$\begin{aligned}
&\left\| \int_s^t e^{(t-u)A} B(X_u) \, du \right\|_{L^p(\mathbb{P};H_\gamma)} \leq \sqrt{\tfrac{p(p-1)}{2} \int_s^t \|e^{(t-u)A} B(X_u)\|^2_{L^p(\mathbb{P};\mathrm{HS}(U,H_\gamma))} \, du} \\
&\leq \sqrt{\tfrac{p(p-1)}{2} \int_s^t \|(-A)^{\eta/2} e^{(t-u)A}\|^2_{L(H)} \|B(X_u)\|^2_{L^p(\mathbb{P};\mathrm{HS}(U,H_{\gamma-\eta/2}))} \, du} \\
&\leq \sqrt{\tfrac{p(p-1)}{2} \int_s^t (t-u)^{-\eta} \left(\|B(0)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})} + b\|X_u\|_{L^p(\mathbb{P};H_\gamma)} \right)^2 \, du} \\
&\leq \left(\|B(0)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})} + b \sup\nolimits_{u\in[0,T]} \|X_u\|_{L^p(\mathbb{P};H_\gamma)} \right) \tfrac{\sqrt{p(p-1)}}{\sqrt{2(1-\eta)}} (t-s)^{(1-\eta)/2}.
\end{aligned} \tag{61}$$

Combining (58)–(61) shows that

$$\begin{aligned}
&\sup\nolimits_{s\in[0,T),t\in(s,T]} \frac{\|(X_t - e^{tA}\xi) - (X_s - e^{sA}\xi)\|_{L^p(\mathbb{P};H_\gamma)}}{(t-s)^\beta} \\
&\leq \left(\|F(0)\|_{H_{\gamma-\eta}} + b \sup\nolimits_{u\in[0,T]} \|X_u\|_{L^p(\mathbb{P};H_\gamma)} \right) \tfrac{2\max\{1,T\}}{1-\eta-\beta} \\
&\quad + \left(\|B(0)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})} + b \sup\nolimits_{u\in[0,T]} \|X_u\|_{L^p(\mathbb{P};H_\gamma)} \right) \tfrac{\sqrt{2p(p-1)\max\{1,T\}}}{\sqrt{1-\eta-2\beta}}.
\end{aligned} \tag{62}$$

The proof of Lemma 5.1 is thus completed. □

Corollary 5.2. *Assume the setting in Section 5.1, assume that* $\sup_{t\in[0,T]} \|X_t\|_{L^p(\mathbb{P};H_\gamma)} < \infty$, *and assume that* $\frac{1}{p} < \frac{(1-\eta)}{2}$. *Then there exist a stochastic process* $Y\colon [0,T]\times\Omega \to H_\gamma$ *with continuous sample paths such that for all* $t \in [0,T]$ *it holds* $\mathbb{P}$*-a.s. that* $X_t = Y_t$.

Proof of Corollary 5.2. Note that Lemma 5.1 combined with the Kolmogorov-Chentsov theorem proves that there exists a modification with continuous sample paths of the stochastic process $[0,T]\times\Omega \ni (t,\omega) \mapsto X_t(\omega) - e^{tA}\xi(\omega) \in H_\gamma$. In addition, observe that the fact that A is the generator of a strongly continuous semigroup implies that the stochastic process $[0,T]\times\Omega \ni (t,\omega) \mapsto e^{tA}\xi(\omega) \in H_\gamma$ has continuous sample paths. The proof of Corollary 5.2 is thus completed. □

6 Convergence of spatial spectral Galerkin discretizations

In this section we establish uniform convergence in probability of spatial spectral Galerkin approximations in the case of SEEs with semi-globally Lipschitz continuous coefficients (cf., e.g., Kurniawan [22]); see Proposition 6.4 below. Proposition 6.4 (and its consequence in Corollary 6.5 respectively) is a tool used in the proof of our main result in Theorem 7.6 below (see Proposition 7.3 below). In our proof of Proposition 6.4 we employ Corollary 2.9 in Cox et al. [4] (which is a generalization of Lemma A1 in Bally et al. [1]) and a nowadays well-known localization procedure (see, e.g., Gyöngy [6] and Printems [25, Lemma 4.8]). There are a number of quite similar results in the literature (see, e.g., Cox et al. [4, Corollary 3.3], Gyöngy [6], Kurniawan [22, Lemma 4.2.2], Printems [25, Lemma 4.8]) and Proposition 6.4 is a minor extension of the results in the literature. The main difference between Proposition 6.4 and known results in the literature is that Proposition 6.4 does only prove convergence in probability with no rate of convergence but Proposition 6.4 does not assume any growth condition of the eigenvalues of the dominant linear operator appearing in the considered SEE; see (63) below. In particular, Proposition 6.4 also applies to SEEs in which the dominant linear operator A in (63) is a bounded linear operator.

6.1 Setting

Assume the setting in Section 1.2, let $b \in [0,\infty)$, $\eta \in [0,1)$, $F \in \mathcal{C}(H_\gamma, H_{\gamma-\eta})$, $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U, H_{\gamma-\eta/2}))$, let $I_n \in \mathcal{P}(\mathbb{H})$, $n \in \mathbb{N}_0$, satisfy $\cup_{n\in\mathbb{N}} \left(\cap_{m\in\{n+1,n+2,\ldots\}} I_m\right) = \mathbb{H} = I_0$, let $P_I \in L(H_{-1})$, $I \in \mathcal{P}(\mathbb{H})$, be the linear operators with the property that for all $x \in H$, $I \in \mathcal{P}(\mathbb{H})$ it holds that $P_I x = \sum_{h\in I} \langle h, x\rangle_H h$, and let $X^n\colon [0,T]\times\Omega \to H_\gamma$, $n \in \mathbb{N}_0$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths such that for all $t \in [0,T]$, $n \in \mathbb{N}_0$ it holds $\mathbb{P}$-a.s. that

$$X^n_t = e^{(t-s)A} P_{I_n}(\xi) + \int_0^t e^{(t-s)A} P_{I_n} F(X^n_s)\, ds + \int_0^t e^{(t-s)A} P_{I_n} B(X^n_s)\, dW_s. \tag{63}$$

6.2 Convergence in the case of globally Lipschitz continuous coefficients

Corollary 6.1. *Assume the setting in Section 6.1, assume that* $\xi \in L^p(\mathbb{P}; H_\gamma)$, *assume that for all* $x, y \in H_\gamma$ *it holds that* $\max\{\|F(x) - F(y)\|_{H_{\gamma-\eta}}, \|B(x) - B(y)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}\} \leq b\|x-y\|_{H_\gamma}$, *let* $\beta \in [0, {}^{(1-\eta)}\!/_2)$, *and assume that for all* $n \in \mathbb{N}$ *it holds that* $\sup_{t\in[0,T]} \|X^n_t\|_{L^p(\mathbb{P};H_\gamma)} < \infty$. *Then*

$$\sup_{n\in\mathbb{N}} \left(\sup_{s\in[0,T),t\in(s,T]} \frac{\|(X^n_t - e^{tA}P_{I_n}\xi) - (X^n_s - e^{sA}P_{I_n}\xi)\|_{L^p(\mathbb{P};H_\gamma)}}{(t-s)^\beta} \right) < \infty. \tag{64}$$

Proof of Corollary 6.1. Note that, e.g., [16, Corollary 6.1.8] shows that $\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|X_t^n\|_{L^p(\mathbb{P};H_\gamma)} < \infty$. Lemma 5.1 hence proves (64). The proof of Corollary 6.1 is thus completed. $\square$

lemma 6.2. *Assume the setting in Section 6.1, assume that for all $x, y \in H_\gamma$ it holds that $\max\{\|F(x)-F(y)\|_{H_{\gamma-\eta}}, \|B(x) - B(y)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}\} \leq b\|x - y\|_{H_\gamma}$, and assume that for all $n \in \mathbb{N}_0$ it holds that $\sup_{t\in[0,T]} \|X_t^n\|_{L^p(\mathbb{P};H_\gamma)} < \infty$. Then*

$$\lim_{n\to\infty} \big(\sup_{t\in[0,T]} \|(X_t^0 - e^{tA}\xi) - (X_t^n - e^{tA}P_{I_n}\xi)\|_{L^p(\mathbb{P};H_\gamma)}\big) = 0 \tag{65}$$

and

$$\lim_{n\to\infty} \big(\sup_{t\in[0,T]} \|X_t^0 - X_t^n\|_{L^p(\mathbb{P};H_\gamma)}\big) = 0. \tag{66}$$

Proof of Lemma 6.2. Let $\mathcal{E}_r\colon [0,\infty) \to [0,\infty), r \in (0,\infty)$, be the functions with the property that for all $x \in [0,\infty), r \in (0,\infty)$ it holds that $\mathcal{E}_r(x) = \big(\sum_{n=0}^{\infty} \frac{(x^2\Gamma(r))^n}{\Gamma(nr+1)}\big)^{1/2}$ (cf., e.g., Henry [9, Section 7.1] and [16, Definition 3.3.1]). Observe that, e.g., [16, Lemma 6.1.7] proves that for all $n \in \mathbb{N}$ it holds that

$$\sup_{t\in[0,T]} \|X_t^0 - X_t^n\|_{L^p(\mathbb{P};H_\gamma)} \leq \sqrt{2} \sup_{t\in[0,T]} \|P_{\mathbb{H}\setminus I_n} X_t^0\|_{L^p(\mathbb{P};H_\gamma)} \mathcal{E}_{1-\eta}\Big(\tfrac{\sqrt{2}T^{1-\eta}b}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}b\Big). \tag{67}$$

Let $J_n \subseteq \mathbb{H}$, $n \in \mathbb{N}$, be the sets with the property that for all $n \in \mathbb{N}$ it holds that $J_n = \cap_{m\in\{n+1,n+2,\dots\}} I_m$. Next, let $f_n\colon [0,T] \to [0,\infty), n \in \mathbb{N}$, be the functions with the property that for all $t \in [0,T], n \in \mathbb{N}$ it holds that $f_n(t) = \|P_{\mathbb{H}\setminus J_n}(X_t^0 - e^{tA}\xi)\|_{L^p(\mathbb{P};H_\gamma)}$. Corollary 6.1 proves that the functions $f_n, n \in \mathbb{N}$, are continuous. Moreover, note that the sequence $(f_n)_{n\in\mathbb{N}}$ is non-increasing. Furthermore, observe that Lebesgue's dominated convergence theorem shows that for all $t \in [0,T]$ it holds that $\lim_{n\to\infty} f_n(t) = 0$. We can thus apply Dini's theorem to obtain that $\lim_{n\to\infty} \sup_{t\in[0,T]} f_n(t) = 0$, i.e., that

$$\lim_{n\to\infty} \big(\sup_{t\in[0,T]} \|P_{\mathbb{H}\setminus J_n}(X_t^0 - e^{tA}\xi)\|_{L^p(\mathbb{P};H_\gamma)}\big) = 0. \tag{68}$$

This proves that

$$\lim_{n\to\infty} \big(\sup_{t\in[0,T]} \|P_{\mathbb{H}\setminus I_n}(X_t^0 - e^{tA}\xi)\|_{L^p(\mathbb{P};H_\gamma)}\big) = 0. \tag{69}$$

Moreover, Lebesgue's theorem of dominated convergence proves that $\lim_{n\to\infty} \|P_{\mathbb{H}\setminus I_n}\xi\|_{L^p(\mathbb{P};H_\gamma)} = 0$. This and the fact that for all $n \in \mathbb{N}$ it holds that $\sup_{t\in[0,T]} \|P_{\mathbb{H}\setminus I_n} e^{tA}\xi\|_{L^p(\mathbb{P};H_\gamma)} \leq \|P_{\mathbb{H}\setminus I_n}\xi\|_{L^p(\mathbb{P};H_\gamma)}$ imply that

$$\lim_{n\to\infty} \big(\sup_{t\in[0,T]} \|P_{\mathbb{H}\setminus I_n} e^{tA}\xi\|_{L^p(\mathbb{P};H_\gamma)}\big) = 0. \tag{70}$$

Combining (67), (69), (70), and the triangle inequality proves (66). Finally, observe that (66), (70), and the triangle inequality prove (65). The proof of Lemma 6.2 is thus completed. $\square$

Corollary 6.3. *Assume the setting in Section 6.1, assume that $p(1-\eta) > 2$, assume that for all $n \in \mathbb{N}_0$ it holds that $\sup_{t\in[0,T]} \|X_t^n\|_{L^p(\mathbb{P};H_\gamma)} < \infty$, and assume that for all $x, y \in H_\gamma$ it holds that $\max\{\|F(x) - F(y)\|_{H_{\gamma-\eta}}, \|B(x) - B(y)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}\} \leq b\|x - y\|_{H_\gamma}$. Then*

$$\lim_{n\to\infty} \Big(\mathbb{E}\Big[\sup_{t\in[0,T]} \|X_t^0 - X_t^n\|_{H_\gamma}^p\Big]\Big) = 0. \tag{71}$$

Proof of Corollary 6.3. First of all, observe that Corollary 6.1, Lemma 6.2, and Corollary 2.9 in Cox et al. [4] (cf. also [1, Lemma A1]) prove that

$$\lim_{n\to\infty}\left(\mathbb{E}\Big[\sup_{t\in[0,T]}\|(X_t^0 - e^{tA}\xi) - (X_t^n - e^{tA}P_{I_n}\xi)\|_{H_\gamma}^p\Big]\right) = 0. \tag{72}$$

Next note that Fatou's lemma shows that

$$\begin{aligned}&\limsup_{n\to\infty}\left(\mathbb{E}\Big[\sup_{t\in[0,T]}\|e^{tA}(P_{\mathbb{H}} - P_{I_n})\xi\|_{H_\gamma}^p\Big]\right) \le \limsup_{n\to\infty}\left(\mathbb{E}\Big[\|(P_{\mathbb{H}} - P_{I_n})\xi\|_{H_\gamma}^p\Big]\right)\\ &\le \mathbb{E}\Big[\limsup_{n\to\infty}\|(P_{\mathbb{H}} - P_{I_n})\xi\|_{H_\gamma}^p\Big] = 0.\end{aligned} \tag{73}$$

Combining (72) and (73) with the triangle inequality proves (71). The proof of Corollary 6.3 is thus completed. □

6.3 Convergence in the case of semi-globally Lipschitz continuous coefficients

Proposition 6.4. *Assume the setting in Section 6.1 and assume that for every $R \in [0,\infty)$ there exists a real number $K \in [0,\infty)$ such that for all $x, y \in H_\gamma$ with $\max\{\|x\|_{H_\gamma}, \|y\|_{H_\gamma}\} \le R$ it holds that $\max\{\|F(x) - F(y)\|_{H_{\gamma-\eta}}, \|B(x) - B(y)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}\} \le K\|x-y\|_{H_\gamma}$. Then for all $\varepsilon \in (0,\infty)$ it holds that*

$$\lim_{n\to\infty}\left(\mathbb{P}\big[\sup_{t\in[0,T]}\|X_t^0 - X_t^n\|_{H_\gamma} \ge \varepsilon\big]\right) = 0. \tag{74}$$

Proof of Proposition 6.4. Throughout this proof let $q \in (2/(1-\eta), \infty)$ be a real number and let $\phi_R\colon \mathbb{R} \to [0,1]$, $R \in (0,\infty)$, be infinitely often differentiable functions such that for all $x \in [-R,R]$ it holds that $\phi_R(x) = 1$ and such that for all $x \in (-\infty,-R-1] \cup [R+1,\infty)$ it holds that $\phi_R(x) = 0$. Moreover, let $F_R\colon H_\gamma \to H_{\gamma-\eta}$, $R \in (0,\infty)$, and $B_R\colon H_\gamma \to \mathrm{HS}(U, H_{\gamma-\eta/2})$, $R \in (0,\infty)$, be the functions with the property that for all $x \in H_\gamma$, $R \in (0,\infty)$ it holds that $F_R(x) = F(x)\phi_R(\|x\|_{H_\gamma})$ and $B_R(x) = B(x)\phi_R(\|x\|_{H_\gamma})$. In the next step we observe that, e.g., Theorem 5.1 in [17] and, e.g., Corollary 5.2 (see also, e.g., Van Neerven et al. [30, Theorem 6.2]) prove that there exist up to modification unique $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes $X^{n,R}\colon [0,T]\times\Omega \to H_\gamma$, $R \in (0,\infty)$, $n \in \mathbb{N}_0$, with continuous sample paths such that for all $R \in (0,\infty)$, $n \in \mathbb{N}_0$ it holds $\sup_{t\in[0,T]}\|X_t^{n,R}\|_{L^q(\mathbb{P};H_\gamma)} < \infty$ and such that for all $t \in [0,T]$, $R \in (0,\infty)$, $n \in \mathbb{N}_0$ it holds $\mathbb{P}$-a.s. that

$$X_t^{n,R} = e^{tA}\mathbb{1}_{\{\|P_{I_n}\xi\|_{H_\gamma}<R\}}P_{I_n}(\xi) + \int_0^t e^{(t-s)A}P_{I_n}F_R(X_s^{n,R})\,ds + \int_0^t e^{(t-s)A}P_{I_n}B_R(X_s^{n,R})\,dW_s. \tag{75}$$

Furthermore, note that for all $x \in H_\gamma$, $R \in [0,\infty)$ with $\|x\|_{H_\gamma} \le R$ it holds that $F_R(x) = F(x)$ and $B_R(x) = B(x)$. Next, let $\tau^{n,R}\colon \Omega \to [0,T]$, $n \in \mathbb{N}_0$, $R \in (0,\infty)$, and $\rho^{n,R}\colon \Omega \to [0,T]$, $n \in \mathbb{N}$, $R \in (0,\infty)$, be the stopping times with the property that for all $n \in \mathbb{N}_0$, $R \in (0,\infty)$ it holds that

$$\tau^{n,R} = \inf\big(\{t \in [0,T]\colon \|X_t^n\|_{H_\gamma} \ge R\} \cup \{T\}\big) \tag{76}$$

and

$$\rho^{n,R} = \inf\big(\{t \in [0,T]\colon \|X_t^0 - X_t^n\|_{H_\gamma} \ge R\} \cup \{T\}\big)\,. \tag{77}$$

Observe that, e.g., Lemma 4.2.2 in Kurniawan [22] and Markov's inequality prove that for all $n \in \mathbb{N}, R \in (0,\infty), q \in (2/(1-\eta),\infty), \varepsilon \in (0,1)$ it holds that

$$\begin{aligned}
&\mathbb{P}\big[\sup\nolimits_{t\in[0,T]} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\big] - \mathbb{P}\big[\sup\nolimits_{t\in[0,T]} \|X_t^0\|_{H_\gamma} \geq R\big] \\
&\leq \mathbb{P}\big[\big\{\sup\nolimits_{t\in[0,T]} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\big\} \cap \{\sup\nolimits_{t\in[0,T]} \|X_t^0\|_{H_\gamma} < R\}\big] \\
&= \mathbb{P}\big[\big\{\sup\nolimits_{t\in[0,\rho^{n,\varepsilon}]} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\big\} \cap \big\{\sup\nolimits_{t\in[0,T]} \|X_t^0\|_{H_\gamma} < R\big\}\big] \\
&= \mathbb{P}\big[\big\{\sup\nolimits_{t\in[0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t,\rho^{n,\varepsilon}\geq t\}} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\big\} \cap \big\{\sup\nolimits_{t\in[0,T]} \|X_t^0\|_{H_\gamma} < R\big\}\big] \\
&= \mathbb{P}\big[\big\{\sup\nolimits_{t\in[0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t,\rho^{n,\varepsilon}\geq t,\tau^{n,R+1}>t\}} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\big\} \cap \big\{\sup\nolimits_{t\in[0,T]} \|X_t^0\|_{H_\gamma} < R\big\}\big] \\
&\leq \mathbb{P}\big[\sup\nolimits_{t\in[0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t,\rho^{n,\varepsilon}\geq t,\tau^{n,R+1}>t\}} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\big] \\
&= \mathbb{P}\Big[\sup\nolimits_{t\in[0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t,\rho^{n,\varepsilon}\geq t,\tau^{n,R+1}>t\}} \big\|X^0_{\min\{t,\tau^{0,R+1}\}} - X^n_{\min\{t,\tau^{n,R+1}\}}\big\|_{H_\gamma} \geq \varepsilon\Big] \\
&= \mathbb{P}\Bigg[\sup_{t\in[0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t,\rho^{n,\varepsilon}\geq t,\tau^{n,R+1}>t\}} \big\|\mathbb{1}_{\{\|\xi\|_{H_\gamma}<R+1\}} X^0_{\min\{t,\tau^{0,R+1}\}} - \mathbb{1}_{\{\|P_{I_n}\xi\|_{H_\gamma}<R+1\}} X^n_{\min\{t,\tau^{n,R+1}\}}\big\|_{H_\gamma} \geq \varepsilon\Bigg] \\
&= \mathbb{P}\Bigg[\sup_{t\in[0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t,\rho^{n,\varepsilon}\geq t,\tau^{n,R+1}>t\}} \big\|\mathbb{1}_{\{\|\xi\|_{H_\gamma}<R+1\}} X^{0,R+1}_{\min\{t,\tau^{0,R+1}\}} - \mathbb{1}_{\{\|P_{I_n}\xi\|_{H_\gamma}<R+1\}} X^{n,R+1}_{\min\{t,\tau^{n,R+1}\}}\big\|_{H_\gamma} \geq \varepsilon\Bigg] \\
&= \mathbb{P}\Big[\sup\nolimits_{t\in[0,\rho^{n,\varepsilon}]} \mathbb{1}_{\{\tau^{0,R}>t,\rho^{n,\varepsilon}\geq t,\tau^{n,R+1}>t\}} \big\|X_t^{0,R+1} - X_t^{n,R+1}\big\|_{H_\gamma} \geq \varepsilon\Big] \\
&\leq \mathbb{P}\Big[\sup\nolimits_{t\in[0,T]} \big\|X_t^{0,R+1} - X_t^{n,R+1}\big\|_{H_\gamma} \geq \varepsilon\Big] \leq \varepsilon^{-q} \cdot \mathbb{E}\Big[\sup\nolimits_{t\in[0,T]} \big\|X_t^{0,R+1} - X_t^{n,R+1}\big\|^q_{H_\gamma}\Big].
\end{aligned} \tag{78}$$

Corollary 6.3 therefore proves that for all $R \in (0,\infty), \varepsilon \in (0,1)$ it holds that

$$\lim_{n\to\infty} \mathbb{P}\big[\sup\nolimits_{t\in[0,T]} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\big] = \mathbb{P}\big[\sup\nolimits_{t\in[0,T]} \|X_t^0\|_{H_\gamma} \geq R\big]. \tag{79}$$

In the next step we let $R \in (0,\infty)$ in (79) tend to ∞ to obtain that for all $\varepsilon \in (0,\infty)$ it holds that $\lim_{n\to\infty} \mathbb{P}\big[\sup_{t\in[0,T]} \|X_t^0 - X_t^n\|_{H_\gamma} \geq \varepsilon\big] = 0$. The proof of Proposition 6.4 is thus completed. ☐

Corollary 6.5. *Assume the setting in Section 6.1, let $q \in (0,p)$, assume that for every $R \in [0,\infty)$ there exists a real number $K \in [0,\infty)$ such that for all $x, y \in H_\gamma$ with $\max\{\|x\|_{H_\gamma}, \|y\|_{H_\gamma}\} \leq R$ it holds that $\max\{\|F(x)-F(y)\|_{H_{\gamma-\eta}}, \|B(x)-B(y)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}\} \leq K\|x-y\|_{H_\gamma}$, and assume that $\limsup_{n\to\infty}\big(\sup_{t\in[0,T]} \|X_t^n\|_{L^p(\mathbb{P};H_\gamma)}\big) < \infty$. Then $\sup_{t\in[0,T]} \|X_t^0\|_{L^p(\mathbb{P};H_\gamma)} < \infty$ and*

$$\lim\nolimits_{n\to\infty}\big(\sup\nolimits_{t\in[0,T]} \|X_t^0 - X_t^n\|_{L^q(\mathbb{P};H_\gamma)}\big) = 0. \tag{80}$$

Proof of Corollary 6.5. Observe that Proposition 6.4 combined with, e.g., Lemma 4.10 in Kurniawan [22] (see also, e.g., [12, Section 3.4.1]) proves $\sup_{t\in[0,T]} \|X_t^0\|_{L^p(\mathbb{P};H_\gamma)} < \infty$ and (80). The proof of Corollary 6.5 is thus competed. ☐

7 Strong convergence rates for full discrete nonlinearities-stopped approximation schemes

In this section we use the results established in Sections 2, 3, 4, and 6 as well as consequences of the perturbation estimate in Theorem 2.10 in Hutzenthaler & Jentzen [11] to prove Theorem 7.6 (the main result of this article).

7.1 Setting

Assume the setting in Section 1.2, let $F \in \mathcal{C}(H_\gamma, H)$, $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U,H))$, $\varepsilon \in (0,\infty)$, assume that $\gamma < \min\{1-\alpha, 1/2 - \beta\}$, let $X\colon [0,T]\times\Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s)\,ds + \int_0^t e^{(t-s)A} B(X_s)\,dW_s, \tag{81}$$

let $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$ be the linear operators with the property that for all $x \in H, I \in \mathcal{P}(\mathbb{H})$ it holds that $P_I(x) = \sum_{h\in I} \langle h, x\rangle_H h$, let $Y^{N,I,R}\colon [0,T]\times\Omega \to P_I(H)$, $N \in \mathbb{N}$, $I \in \mathcal{P}_0(\mathbb{H})$, $R \in L(U)$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes such that for all $t \in [0,T]$, $N \in \mathbb{N}$, $I \in \mathcal{P}_0(\mathbb{H})$, $R \in L(U)$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} Y_t^{N,I,R} &= e^{tA}P_I\xi + \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\,\mathbb{1}_{\left\{\|P_I F(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}})\|_H + \|P_I B(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)} \le \left(\frac{N}{T}\right)^\theta\right\}} P_I F(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}})\,ds \\ &\quad + \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\,\mathbb{1}_{\left\{\|P_I F(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}})\|_H + \|P_I B(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}})\|_{HS(U,H)} \le \left(\frac{N}{T}\right)^\theta\right\}} P_I B(Y^{N,I,R}_{\lfloor s\rfloor_{T/N}}) R\,dW_s, \end{aligned} \tag{82}$$

and assume that for all $x,y \in H_\gamma$ it holds that $\max\{\|F(x)-F(y)\|_H, \|B(x)-B(y)\|_{\mathrm{HS}(U,H)}\} \le C\,\|x-y\|_{H_\delta}(1 + \|x\|_{H_\gamma}^c + \|y\|_{H_\gamma}^c)$ and $\max\{\|F(x)\|_{H_{-\alpha}}, \|B(x)\|_{\mathrm{HS}(U,H_{-\beta})}\} \le C(1+\|x\|_H^a)$.

7.2 Strong convergence rates for space discretizations

lemma 7.1. *Assume the setting in Section 7.1, let $I_1, I_2 \in \mathcal{P}_0(\mathbb{H})$, $q, r \in [1,\infty]$ satisfy $I_1 \subseteq I_2$ and $\frac{1}{q}+\frac{1}{r} = 1$, assume that for all $x, y \in H_1$ it holds that $\langle x-y, Ax-Ay+F(x)-F(y)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|B(x)-B(y)\|^2_{\mathrm{HS}(U,H)} \le C\|x-y\|_H^2$, and let $X^{I_k}\colon [0,T]\times\Omega \to P_{I_k}(H_\gamma)$, $k \in \{1,2\}$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths such that for all $s \in [0,T]$, $k \in \{1,2\}$ it holds $\mathbb{P}$-a.s. that*

$$X_s^{I_k} = P_{I_k}(\xi) + \int_0^s [AX_u^{I_k} + P_{I_k}F(X_u^{I_k})]\,du + \int_0^s P_{I_k}B(X_u^{I_k})\,dW_u. \tag{83}$$

Then

$$\begin{aligned} &\sup\nolimits_{t\in[0,T]} \|X_t^{I_1} - X_t^{I_2}\|_{L^p(\mathbb{P};H)} \\ &\le \left(\|(-A)^{-\delta}\|_{L(H)} + e^{(C+1)T}Cp(1+\tfrac{1}{\varepsilon})\right) \sup_{u\in[0,T]} \|P_{\mathbb{H}\setminus I_1} X_u^{I_2}\|_{L^{qp}(\mathbb{P};H_\delta)} \left[1 + 2\sup_{u\in[0,T]} \|X_u^{I_2}\|^c_{L^{rpc}(\mathbb{P};H_\gamma)}\right]. \end{aligned} \tag{84}$$

Proof of Lemma 7.1. We intend to prove Lemma 7.1 through an application of Proposition 3.6 in Hutzenthaler & Jentzen [11]. For this we now check the assumptions in Proposition 3.6 in Hutzenthaler & Jentzen [11]. Note that for all $x \in H_\gamma$, $y \in P_{I_1}(H)$ it holds that

$$\begin{aligned} &\langle P_{I_1}x - y, P_{I_1}[AP_{I_1}x + F(P_{I_1}x)] - P_{I_1}[Ay + F(y)]\rangle_H + \tfrac{(p-1)(1+\varepsilon)}{2}\|P_{I_1}[B(P_{I_1}x) - B(y)]\|^2_{\mathrm{HS}(U,H)} \\ &\le \langle P_{I_1}x - y, P_{I_1}[AP_{I_1}x + F(P_{I_1}x)] - P_{I_1}[Ay + F(y)]\rangle_H + \tfrac{(p-1)(1+\varepsilon)}{2}\|B(P_{I_1}x) - B(y)\|^2_{\mathrm{HS}(U,H)} \\ &= \langle P_{I_1}x - y, AP_{I_1}x + F(P_{I_1}x) - Ay - F(y)]\rangle_H + \tfrac{(p-1)(1+\varepsilon)}{2}\|B(P_{I_1}x) - B(y)\|^2_{\mathrm{HS}(U,H)} \\ &\le C\|P_{I_1}x - y\|_H^2. \end{aligned} \tag{85}$$

Moreover, observe that for all $x \in H_\gamma, y \in P_{I_1}(H)$ it holds that

$$\begin{aligned}
&\langle y - P_{I_1}x, P_{I_1}[AP_{I_1}x + F(P_{I_1}x)] - P_{I_1}[Ax + F(x)]\rangle_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P_{I_1}[B(P_{I_1}x) - B(x)]\|^2_{\mathrm{HS}(U,H)} \\
&= \langle y - P_{I_1}x, AP_{I_1}x + F(P_{I_1}x) - AP_{I_1}x - F(x)\rangle_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P_{I_1}[B(P_{I_1}x) - B(x)]\|^2_{\mathrm{HS}(U,H)} \\
&\leq \|y - P_{I_1}x\|_H\|F(P_{I_1}x) - F(x)\|_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|B(P_{I_1}x) - B(x)\|^2_{\mathrm{HS}(U,H)} \\
&\leq \tfrac{1}{2}\|y - P_{I_1}x\|^2_H + \tfrac{1}{2}\|F(P_{I_1}x) - F(x)\|^2_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|B(P_{I_1}x) - B(x)\|^2_{\mathrm{HS}(U,H)} \\
&\leq \tfrac{1}{2}\|y - P_{I_1}x\|^2_H + \tfrac{1}{2}\Big(C\sqrt{1 + (p-1)(1 + 1/\varepsilon)}\|P_{I_1}x - x\|_{H_\delta}(1 + \|P_{I_1}x\|^c_{H_\gamma} + \|x\|^c_{H_\gamma})\Big)^2.
\end{aligned} \tag{86}$$

Next observe that the Hölder inequality proves that

$$\begin{aligned}
&\big\|\|P_{I_1}X^{I_2} - X^{I_2}\|_{H_\delta}(1 + \|P_{I_1}X^{I_2}\|^c_{H_\gamma} + \|X^{I_2}\|^c_{H_\gamma})\big\|_{L^p(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \\
&\leq \big\|\|P_{I_1}X^{I_2} - X^{I_2}\|_{H_\delta}\big\|_{L^{qp}(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})}\big\|1 + \|P_{I_1}X^{I_2}\|^c_{H_\gamma} + \|X^{I_2}\|^c_{H_\gamma}\big\|_{L^{rp}(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \\
&\leq T^{1/p}\sup_{u\in[0,T]}\|P_{\mathbb{H}\setminus I_1}X^{I_2}_u\|_{L^{qp}(\mathbb{P};H_\delta)}\left[1 + \sup_{u\in[0,T]}\|P_{I_1}X^{I_2}_u\|^c_{L^{rpc}(\mathbb{P};H_\gamma)} + \sup_{u\in[0,T]}\|X^{I_2}_u\|^c_{L^{rpc}(\mathbb{P};H_\gamma)}\right].
\end{aligned} \tag{87}$$

Combining (85)–(87) with Proposition 3.6 in Hutzenthaler & Jentzen [11] shows that for all $t \in [0,T]$ it holds that

$$\begin{aligned}
&\|X^{I_1}_t - X^{I_2}_t\|_{L^p(\mathbb{P};H)} \leq e^{\frac{1}{2}-\frac{1}{p}+(C+\frac{1}{2})T}C\sqrt{T\big(1 + (p-1)(1 + \tfrac{1}{\varepsilon})\big)}\sup_{u\in[0,T]}\|P_{\mathbb{H}\setminus I_1}X^{I_2}_u\|_{L^{qp}(\mathbb{P};H_\delta)} \\
&\cdot\big[1 + \sup_{u\in[0,T]}\|P_{I_1}X^{I_2}_u\|^c_{L^{rpc}(\mathbb{P};H_\gamma)} + \sup_{u\in[0,T]}\|X^{I_2}_u\|^c_{L^{rpc}(\mathbb{P};H_\gamma)}\big] + \sup_{u\in[0,T]}\|P_{\mathbb{H}\setminus I_1}X^{I_2}_u\|_{L^p(\mathbb{P};H)}.
\end{aligned} \tag{88}$$

The proof of Lemma 7.1 is thus completed. □

7.3 Strong convergence rates for noise discretizations

lemma 7.2. *Assume the setting in Section 7.1, let $\kappa \in (0,\infty), \nu \in (0, 1/2 - \delta), \eta \in [\gamma,\infty), I \in \mathcal{P}_0(\mathbb{H}), R_1, R_2 \in L(U)$, assume that for all $x, y \in H_1$ it holds that $\langle x - y, Ax - Ay + F(x) - F(y)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|B(x) - B(y)\|^2_{\mathrm{HS}(U,H)} \leq C\|x - y\|^2_H$, and let $X^{I,R_k}\colon [0,T] \times \Omega \to P_I(H_\gamma), k \in \{1,2\}$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths such that for all $t \in [0,T], k \in \{1,2\}$ it holds $\mathbb{P}$-a.s. that*

$$X^{I,R_k}_t = P_I(\xi) + \int_0^t [AX^{I,R_k}_s + P_IF(X^{I,R_k}_s)]\,ds + \int_0^t P_IB(X^{I,R_k}_s)R_k\,dW_s. \tag{89}$$

Then

$$\begin{aligned}
&\sup_{t\in[0,T]}\|X^{I,R_1}_t - X^{I,R_2}_t\|_{L^p(\mathbb{P};H)} \\
&\leq \tfrac{\max\{1,T^{1/2}\}\sqrt{p(2p-1)}}{\sqrt{1-2(\delta+\nu)}}\left[\sup_{v\in H_\eta}\tfrac{\|B(v)(R_2-R_1)\|_{\mathrm{HS}(U,H_{-\nu})}}{1+\|v\|^\kappa_{H_\eta}}\right]\Big(1 + \sup_{s\in[0,T]}\|X^{I,R_2}_s\|^\kappa_{L^{2p\kappa}(\mathbb{P};H_\eta)}\Big) \\
&\quad\cdot\left(1 + \big(TC^2\max\{1, \|R_1\|^2_{L(U)}\}p(1 + \tfrac{1}{\varepsilon})\big)^{\frac{1}{p}}\exp\!\left(\tfrac{TC^2\max\{1,\|R_1\|^2_{L(U)}\}p(1+1/\varepsilon)}{2}\right)\right) \\
&\quad\cdot\Bigg[3\big(1 + \|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)}\big)C\max\{1,T\}\max\{1, \|R_1\|_{L(U)}, \|R_2\|_{L(U)}\} \\
&\quad\cdot\left[\tfrac{1}{1-(\gamma+\alpha)} + \sqrt{\tfrac{pc(2pc-1)}{1-2(\gamma+\beta)}}\right]\left[1 + \sup_{t\in[0,T]}\|X^{I,R_2}_t\|^a_{L^{2pca}(\mathbb{P};H)}\right]\Bigg]^c.
\end{aligned} \tag{90}$$

Proof of Lemma 7.2. Throughout this proof let $\chi \in [0, \infty)$ be the real number given by $\chi = \frac{C(p-1)}{p}\big(1 + \frac{C\|R_1\|^2_{L(U)}(p-2)(1+1/\varepsilon)}{2}\big)$. Let $\hat{X}\colon [0,T] \times \Omega \to P_I(H_\gamma)$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$\hat{X}_t = P_I\xi + \int_0^t [A\hat{X}_s + P_I F(X_s^{I,R_2})]\, ds + \int_0^t P_I B(X^{I,R_2}) R_1 \, dW_s. \tag{91}$$

Next observe that Corollary 2.11 in Hutzenthaler & Jentzen [11] combined with the Cauchy-Schwarz inequality proves that for all $t \in [0,T]$ it holds that

$$\begin{aligned}
&\|X_t^{I,R_2} - X_t^{I,R_1}\|_{L^p(\mathbb{P};H)} \\
&\leq \|\hat{X}_t - X_t^{I,R_2}\|_{L^p(\mathbb{P};H)} + e^{(C+\chi)T} \bigg\| p\|X^{I,R_1} - \hat{X}\|_H^{p-2}\Big[\|X^{I,R_1} - \hat{X}\|_H \|P_I F(\hat{X}) - P_I F(X^{I,R_2})\|_H \\
&\quad + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P_I B(X^{I,R_2})R_1 - P_I B(\hat{X})R_1\|^2_{\mathrm{HS}(U,H)} - \chi\|X^{I,R_1} - \hat{X}\|_H^2\Big]^+\bigg\|^{1/p}_{L^1(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})}.
\end{aligned} \tag{92}$$

In addition, note that for all $s \in [0,T]$ it holds that

$$\|P_I F(\hat{X}_s) - P_I F(X_s^{I,R_2})\|_H \leq C\|\hat{X}_s - X_s^{I,R_2}\|_{H_\delta}(1 + \|\hat{X}_s\|^c_{H_\gamma} + \|X_s^{I,R_2}\|^c_{H_\gamma}) \tag{93}$$

and

$$\big\|P_I\big(B(X_s^{I,R_2}) - B(\hat{X}_s)\big)R_1\big\|_{\mathrm{HS}(U,H)} \leq C\|R_1\|_{L(U)}\|X_s^{I,R_2} - \hat{X}_s\|_{H_\delta}(1 + \|X_s^{I,R_2}\|^c_{H_\gamma} + \|\hat{X}_s\|^c_{H_\gamma}). \tag{94}$$

Combining (93) and (94) shows that

$$\begin{aligned}
&\bigg\| p\|X^{I,R_1} - \hat{X}\|_H^{p-2}\Big[\|X^{I,R_1} - \hat{X}\|_H \|P_I F(\hat{X}) - P_I F(X^{I,R_2})\|_H \\
&\quad + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P_I B(X^{I,R_2})R_1 - P_I B(\hat{X})R_1\|^2_{\mathrm{HS}(U,H)} - \chi\|X^{I,R_1} - \hat{X}\|_H^2\Big]^+\bigg\|^{1/p}_{L^1(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \\
&\leq \bigg\| p\|X^{I,R_1} - \hat{X}\|_H^{p-2}\Big[\|X^{I,R_1} - \hat{X}\|_H C\|\hat{X} - X^{I,R_2}\|_{H_\delta}(1 + \|\hat{X}\|^c_{H_\gamma} + \|X^{I,R_2}\|^c_{H_\gamma}) \\
&\quad + \tfrac{(p-1)(1+1/\varepsilon)C^2\|R_1\|^2_{L(U)}}{2}\|X^{I,R_2} - \hat{X}\|^2_{H_\delta}(1 + \|X^{I,R_2}\|^c_{H_\gamma} + \|\hat{X}\|^c_{H_\gamma})^2 - \chi\|X^{I,R_1} - \hat{X}\|_H^2\Big]^+\bigg\|^{1/p}_{L^1(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})}.
\end{aligned} \tag{95}$$

Moreover, observe that Young's inequality proves that for all $s \in [0,T]$ it holds that

$$\begin{aligned}
&\|X_s^{I,R_1} - \hat{X}_s\|_H^{p-1}\|\hat{X}_s - X_s^{I,R_2}\|_{H_\delta}(1 + \|\hat{X}_s\|^c_{H_\gamma} + \|X_s^{I,R_2}\|^c_{H_\gamma}) \\
&\leq \tfrac{p-1}{p}\|X_s^{I,R_1} - \hat{X}_s\|_H^p + \tfrac{1}{p}\|\hat{X}_s - X_s^{I,R_2}\|^p_{H_\delta}(1 + \|\hat{X}_s\|^c_{H_\gamma} + \|X_s^{I,R_2}\|^c_{H_\gamma})^p
\end{aligned} \tag{96}$$

and

$$\begin{aligned}
&\|X_s^{I,R_1} - \hat{X}_s\|_H^{p-2}\|\hat{X}_s - X_s^{I,R_2}\|^2_{H_\delta}(1 + \|\hat{X}_s\|^c_{H_\gamma} + \|X_s^{I,R_2}\|^c_{H_\gamma})^2 \\
&\leq \tfrac{p-2}{p}\|X_s^{I,R_1} - \hat{X}_s\|_H^p + \tfrac{2}{p}\|\hat{X}_s - X_s^{I,R_2}\|^p_{H_\delta}(1 + \|\hat{X}_s\|^c_{H_\gamma} + \|X_s^{I,R_2}\|^c_{H_\gamma})^p.
\end{aligned} \tag{97}$$

Combining (95), (96), and (97) shows that

$$
\begin{aligned}
&\Big\| p\|X^{I,R_1}-\hat{X}\|_H^{p-2}\Big[\|X^{I,R_1}-\hat{X}\|_H\|P_IF(\hat{X})-P_IF(X^{I,R_2})\|_H \\
&\quad+\tfrac{(p-1)(1+1/\varepsilon)}{2}\|P_IB(X^{I,R_2})R_1-P_IB(\hat{X})R_1\|^2_{\mathrm{HS}(U,H)}-\chi\|X^{I,R_1}-\hat{X}\|_H^2\Big]^+\Big\|^{1/p}_{L^1(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \\
&\leq \Big\|(p-1)C\|X^{I,R_1}-\hat{X}\|_H^p+C\|\hat{X}-X^{I,R_2}\|^p_{H_\delta}(1+\|\hat{X}\|^c_{H_\gamma}+\|X^{I,R_2}\|^c_{H_\gamma})^p \\
&\quad+\tfrac{(p-1)(p-2)(1+\frac{1}{\varepsilon})C^2\|R_1\|^2_{L(U)}}{2}\|X^{I,R_1}-\hat{X}\|_H^p \\
&\quad+C^2\|R_1\|^2_{L(U)}(p-1)(1+\tfrac{1}{\varepsilon})\|\hat{X}-X^{I,R_2}\|^p_{H_\delta}(1+\|\hat{X}\|^c_{H_\gamma}+\|X^{I,R_2}\|^c_{H_\gamma})^p \\
&\quad-(p-1)C\Big(1+\tfrac{\|R_1\|^2_{L(U)}C(p-2)(1+1/\varepsilon)}{2}\Big)\|X^{I,R_1}-\hat{X}\|_H^p\Big\|^{1/p}_{L^1(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \\
&=\big(C+C^2\|R_1\|^2_{L(U)}(p-1)(1+\tfrac{1}{\varepsilon})\big)^{\frac{1}{p}}\Big\|\|\hat{X}-X^{I,R_2}\|_{H_\delta}(1+\|\hat{X}\|^c_{H_\gamma}+\|X^{I,R_2}\|^c_{H_\gamma})\Big\|_{L^p(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})}.
\end{aligned}
\tag{98}
$$

Furthermore, note that the Hölder inequality implies that

$$
\begin{aligned}
&\Big\|\|\hat{X}-X^{I,R_2}\|_{H_\delta}(1+\|\hat{X}\|^c_{H_\gamma}+\|X^{I,R_2}\|^c_{H_\gamma})\Big\|_{L^p(\mu_{[0,T]}\otimes\mathbb{P};\mathbb{R})} \\
&\leq T^{\frac{1}{p}}\sup_{t\in[0,T]}\|\hat{X}_t-X_t^{I,R_2}\|_{L^{2p}(\mathbb{P};H_\delta)}\Big(1+\sup_{t\in[0,T]}\|\hat{X}_t\|^c_{L^{2pc}(\mathbb{P};H_\gamma)}+\sup_{t\in[0,T]}\|X_t^{I,R_2}\|^c_{L^{2pc}(\mathbb{P};H_\gamma)}\Big).
\end{aligned}
\tag{99}
$$

Combining (92), (98), and (99) proves that

$$
\begin{aligned}
&\sup\nolimits_{t\in[0,T]}\|X_t^{I,R_2}-X_t^{I,R_1}\|_{L^p(\mathbb{P};H)} \\
&\leq \sup\nolimits_{t\in[0,T]}\|\hat{X}_t-X_t^{I,R_2}\|_{L^p(\mathbb{P};H)}+e^{(C+\chi)T}\big(TC+TC^2\|R_1\|^2_{L(U)}(p-1)(1+\tfrac{1}{\varepsilon})\big)^{\frac{1}{p}} \\
&\cdot\sup\nolimits_{t\in[0,T]}\|\hat{X}_t-X_t^{I,R_2}\|_{L^{2p}(\mathbb{P};H_\delta)}\Big(1+\sup\nolimits_{t\in[0,T]}\|\hat{X}_t\|^c_{L^{2pc}(\mathbb{P};H_\gamma)}+\sup\nolimits_{t\in[0,T]}\|X_t^{I,R_2}\|^c_{L^{2pc}(\mathbb{P};H_\gamma)}\Big).
\end{aligned}
\tag{100}
$$

In the next step observe that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [5] shows that for all $t\in[0,T], r\in[0,\gamma], q\in[2,\infty)$ it holds that

$$
\begin{aligned}
\|\hat{X}_t-X_t^{I,R_2}\|_{L^q(\mathbb{P};H_r)}&=\Big\|\int_0^t e^{(t-s)A}P_IB(X_s^{I,R_2})(R_2-R_1)\,dW_s\Big\|_{L^q(\mathbb{P};H_r)} \\
&\leq\sqrt{\tfrac{q(q-1)}{2}\int_0^t\Big\|e^{(t-s)A}P_IB(X_s^{I,R_2})(R_2-R_1)\Big\|^2_{L^q(\mathbb{P};\mathrm{HS}(U,H_r))}\,ds}.
\end{aligned}
\tag{101}
$$

Moreover, note that, e.g., [16, Theorem 2.5.34] implies that for all $r\in[0,\gamma], t\in[0,T], q\in[2,\infty)$ it

holds that

$$
\begin{aligned}
&\sqrt{\int_0^t \left\| e^{(t-s)A} P_I B(X_s^{I,R_2})(R_2 - R_1) \right\|^2_{L^q(\mathbb{P};\mathrm{HS}(U,H_r))} ds} \\
&\leq \sqrt{\int_0^t \|(-A)^{r+\nu} e^{(t-s)A}\|^2_{L(H)} \|(-A)^{-\nu} B(X_s^{I,R_2})(R_2 - R_1)\|^2_{L^q(\mathbb{P};\mathrm{HS}(U,H))} ds} \\
&\leq \sqrt{\int_0^t (t-s)^{-2(r+\nu)} \|B(X_s^{I,R_2})(R_2 - R_1)\|^2_{L^q(\mathbb{P};\mathrm{HS}(U,H_{-\nu}))} ds} \\
&\leq \tfrac{T^{1/2-(r+\nu)}}{\sqrt{1-2(r+\nu)}} \sup\nolimits_{s\in[0,T]} \|B(X_s^{I,R_2})(R_2 - R_1)\|_{L^q(\mathbb{P};\mathrm{HS}(U,H_{-\nu}))} \\
&\leq \tfrac{T^{1/2-(r+\nu)}}{\sqrt{1-2(r+\nu)}} \left(\sup_{v\in H_\eta} \tfrac{\|B(v)(R_2-R_1)\|_{\mathrm{HS}(U,H_{-\nu})}}{1+\|v\|^\kappa_{H_\eta}} \right) \left(1 + \sup\nolimits_{s\in[0,T]} \|X_s^{I,R_2}\|^\kappa_{L^{q\kappa}(\mathbb{P};H_\eta)} \right).
\end{aligned}
\tag{102}
$$

In addition, note that Lemma 3.1 shows that

$$
\begin{aligned}
&\max\{\sup\nolimits_{t\in[0,T]} \|\hat{X}_t\|_{L^{2pc}(\mathbb{P};H_\gamma)}, \sup\nolimits_{t\in[0,T]} \|X_t^{I,R_2}\|_{L^{2pc}(\mathbb{P};H_\gamma)}\} \leq \|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C\max\{1,T\} \\
&\cdot \max\{1, \|R_1\|_{L(U)}, \|R_2\|_{L(U)}\} \left[\tfrac{1}{1-(\gamma+\alpha)} + \sqrt{\tfrac{pc(2pc-1)}{1-2(\gamma+\beta)}} \right] \left[1 + \sup\nolimits_{t\in[0,T]} \|X_t^{I,R_2}\|^a_{L^{2pca}(\mathbb{P};H)} \right].
\end{aligned}
\tag{103}
$$

Combining (100)-(103) proves that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|X_t^{I,R_2} - X_t^{I,R_1}\|_{L^p(\mathbb{P};H)} \\
&\leq \tfrac{\sqrt{p(2p-1)}T^{1/2-\nu}}{\sqrt{1-2\nu}} \left(\sup_{v\in H_\eta} \tfrac{\|B(v)(R_2-R_1)\|_{\mathrm{HS}(U,H_{-\nu})}}{1+\|v\|^\kappa_{H_\eta}} \right) \left(1 + \sup\nolimits_{s\in[0,T]} \|X_s^{I,R_2}\|^\kappa_{L^{p\kappa}(\mathbb{P};H_\eta)} \right) \\
&+ \tfrac{\sqrt{p(2p-1)} e^{(C+\chi)T} \left(\max\{1, \|R_1\|^2_{L(U)}\} T C^2 p (1+1/\varepsilon) \right)^{1/p} T^{1/2-(\delta+\nu)}}{\sqrt{1-2(\delta+\nu)}} \left(\sup_{v\in H_\eta} \tfrac{\|B(v)(R_2-R_1)\|_{\mathrm{HS}(U,H_{-\nu})}}{1+\|v\|^\kappa_{H_\eta}} \right) \\
&\cdot \left(1 + \sup\nolimits_{s\in[0,T]} \|X_s^{I,R_2}\|^\kappa_{L^{2p\kappa}(\mathbb{P};H_\eta)} \right) \Bigg(1 + 2 \Bigg[\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C\max\{1,T\} \max\{1, \|R_1\|_{L(U)}, \|R_2\|_{L(U)}\} \\
&\cdot \left[\tfrac{1}{1-(\gamma+\alpha)} + \sqrt{\tfrac{pc(2pc-1)}{1-2(\gamma+\beta)}} \right] \left[1 + \sup\nolimits_{t\in[0,T]} \|X_t^{I,R_2}\|^a_{L^{2pca}(\mathbb{P};H)} \right] \Bigg]^c \Bigg).
\end{aligned}
\tag{104}
$$

The proof of Lemma 7.2 is thus completed. ☐

7.4 Strong convergence rates for space-time-noise discretizations

Proposition 7.3. *Assume the setting in Section 7.1, let $q \in (0,p)$, assume that $\xi \in L^{pa}(\mathbb{P}; H_\gamma)$, assume that for all $x \in H_1$ it holds that $\langle x, F(x)\rangle_H + \frac{pa-1}{2}\|B(x)\|^2_{\mathrm{HS}(U,H)} \leq C(1 + \|x\|^2_H)$, let $I_n \in \mathcal{P}_0(\mathbb{H}), n \in \mathbb{N}$, satisfy $\cup_{n\in\mathbb{N}} \left(\cap_{m\in\{n+1,n+2,\dots\}} I_m \right) = \mathbb{H}$, and let $X^I \colon [0,T]\times\Omega \to P_I(H_\gamma)$, $I \in \mathcal{P}_0(\mathbb{H})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths such that for all $t \in [0,T], I \in \mathcal{P}_0(\mathbb{H})$ it holds $\mathbb{P}$-a.s. that*

$$
X_t^I = P_I\xi + \int_0^t [AX_s^I + P_I F(X_s^I)]\, ds + \int_0^t P_I B(X_s^I)\, dW_s.
\tag{105}
$$

Then it holds that $\lim_{n\to\infty}\big(\sup_{t\in[0,T]}\|X_t - X_t^{I_n}\|_{L^q(\mathbb{P};H_\gamma)}\big) = 0$ *and*

$$\sup_{\substack{R\in L^1(U), I\in\mathcal{P}_0(\mathbb{H}),\\ N\in\mathbb{N}, t\in[0,T]}} \mathbb{E}\Big[\big\|Y_t^{N,I,R}\big\|_H^{pa} + \|X_t^I\|_H^{pa} + \big\|Y_t^{N,I,R}\big\|_{H_\gamma}^p + \|X_t\|_{H_\gamma}^p + \|X_t^I\|_{H_\gamma}^p\Big] < \infty. \tag{106}$$

Proof of Proposition 7.3. Itô's formula and Young's inequality prove that for all $t\in[0,T], I\in\mathcal{P}_0(\mathbb{H})$ it holds that

$$\begin{aligned}
&\mathbb{E}[\|X_t^I\|_H^{pa}] \\
&\leq \mathbb{E}[\|X_0^I\|_H^{pa}] + pa\int_0^t \mathbb{E}\Big[\|X_s^I\|_H^{pa-2}\big(\langle X_s^I, P_I F(X_s^I)\rangle_H + \tfrac{pa-1}{2}\|P_I B(X_s^I)\|^2_{\mathrm{HS}(U,H)}\big)\Big]\,ds \\
&\leq \mathbb{E}[\|X_0^I\|_H^{pa}] + paC\int_0^t \mathbb{E}\big[\|X_s^I\|_H^{pa-2}(1+\|X_s^I\|_H^2)\big]\,ds \\
&\leq \mathbb{E}[\|X_0^I\|_H^{pa}] + 2C\int_0^t \mathbb{E}\big[(pa-1)\|X_s^I\|_H^{pa} + 1\big]ds.
\end{aligned} \tag{107}$$

Therefore, Gronwall's lemma proves that for all $t\in[0,T], I\in\mathcal{P}_0(\mathbb{H})$ it holds that $\mathbb{E}[\|X_t^I\|_H^{pa}] \leq (\mathbb{E}[\|\xi\|_H^{pa}] + 2CT)\,e^{2C(pa-1)T}$. Lemma 3.1 hence implies that $\sup_{I\in\mathcal{P}_0(\mathbb{H})}\sup_{t\in[0,T]}\|X_t^I\|_{L^p(\mathbb{P};H_\gamma)} < \infty$. Combining this with Corollary 6.5 shows that $\lim_{n\to\infty}\big(\sup_{t\in[0,T]}\|X_t - X_t^{I_n}\|_{L^q(\mathbb{P};H_\gamma)}\big) = 0$ and that $\sup_{t\in[0,T]}\|X_t\|_{L^p(\mathbb{P};H_\gamma)} < \infty$. Moreover, note that combining Lemma 2.2 and Lemma 3.1 proves that

$$\sup_{R\in L^1(U)}\sup_{I\in\mathcal{P}_0(\mathbb{H})}\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\Big(\|Y_t^{N,I,R}\|_{L^{pa}(\mathbb{P};H)} + \|Y_t^{N,I,R}\|_{L^p(\mathbb{P};H_\gamma)}\Big) < \infty. \tag{108}$$

The proof of Proposition 7.3 is thus completed. $\square$

Corollary 7.4. *Assume the setting in Section 7.1, assume that* $\xi\in L^{p(c+1)a}(\mathbb{P};H_\gamma)$, *assume that for all* $x\in H_1$ *it holds that* $\langle x, F(x)\rangle_H + \frac{p(c+1)a-1}{2}\|B(x)\|^2_{\mathrm{HS}(U,H)} \leq C(1+\|x\|_H^2)$, *and let* $X^I\colon[0,T]\times\Omega\to P_I(H_\gamma)$, $I\in\mathcal{P}_0(\mathbb{H})$, *be* $(\mathcal{F}_t)_{t\in[0,T]}$*-adapted stochastic processes with continuous sample paths such that for all* $t\in[0,T], I\in\mathcal{P}_0(\mathbb{H})$ *it holds* $\mathbb{P}$*-a.s. that*

$$X_t^I = P_I\xi + \int_0^t [AX_s^I + P_I F(X_s^I)]\,ds + \int_0^t P_I B(X_s^I)\,dW_s. \tag{109}$$

Then

$$\sup_{\substack{I\in\mathcal{P}_0(\mathbb{H}), N\in\mathbb{N},\\ R\in L^1(U), t\in[0,T]}} \Big\|\|F(Y_t^{N,I,R})\|_H + \|F(X_t)\|_H + \|B(Y_t^{N,I,R})\|_{\mathrm{HS}(U,H)} + \|B(X_t)\|_{\mathrm{HS}(U,H)}\Big\|_{L^p(\mathbb{P};\mathbb{R})} < \infty. \tag{110}$$

Proof of Corollary 7.4. Observe that for all $t\in[0,T], N\in\mathbb{N}, I\in\mathcal{P}_0(\mathbb{H}), R\in L^1(U)$ it holds that

$$\begin{aligned}
&\max\{\|F(Y_t^{N,I,R})\|_{L^p(\mathbb{P};H)}, \|B(Y_t^{N,I,R})\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))}\} \\
&\leq \|F(0)\|_H + \|B(0)\|_{\mathrm{HS}(U,H)} + C\|(-A)^{\delta-\gamma}\|_{L(H)}\big(1 + \|Y_t^{N,I,R}\|_{L^{p(c+1)}(\mathbb{P};H_\gamma)}\big)^{c+1}.
\end{aligned} \tag{111}$$

Proposition 7.3 hence proves that

$$\sup_{R\in L^1(U)}\sup_{I\in\mathcal{P}_0(\mathbb{H})}\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\Big(\|F(Y_t^{N,I,R})\|_{L^p(\mathbb{P};H)} + \|B(Y_t^{N,I,R})\|_{L^p(\mathbb{P};H)}\Big) < \infty. \tag{112}$$

Moreover, note that for all $t \in [0,T]$ it holds that

$$\begin{aligned}
&\max\{\|F(X_t)\|_{L^p(\mathbb{P};H)}, \|B(X_t)\|_{L^p(\mathbb{P};\mathrm{HS}(U,H))}\} \\
&\leq \|F(0)\|_H + \|B(0)\|_{\mathrm{HS}(U,H)} + C\|(-A)^{\delta-\gamma}\|_{L(H)}\big(1 + \|X_t\|_{L^{p(c+1)}(\mathbb{P};H_\gamma)}\big)^{c+1}.
\end{aligned} \tag{113}$$

Proposition 7.3 hence shows that

$$\sup\nolimits_{t\in[0,T]} \|F(X_t)\|_{L^p(\mathbb{P};H)} + \sup\nolimits_{t\in[0,T]} \|B(X_t)\|_{L^p(\mathbb{P};H)} < \infty. \tag{114}$$

The proof of Corollary 7.4 is thus completed. □

Corollary 7.5. *Assume the setting in Section 7.1, let $\eta \in [\gamma, 1/2)$, assume that $\xi(\Omega) \subseteq H_\eta$, assume that $\mathbb{E}\big[\|\xi\|_{H_\eta}^{p(c+1)a}\big] < \infty$, assume that for all $x \in H_1$ it holds that $\langle x, F(x)\rangle_H + \frac{p(c+1)a-1}{2}\|B(x)\|^2_{\mathrm{HS}(U,H)} \leq C(1 + \|x\|^2_H)$, and let $X^I\colon [0,T]\times\Omega \to P_I(H_\gamma)$, $I \in \mathcal{P}_0(\mathbb{H})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths such that for all $t \in [0,T]$, $I \in \mathcal{P}_0(\mathbb{H})$ it holds $\mathbb{P}$-a.s. that*

$$X_t^I = P_I\xi + \int_0^t [AX_s^I + P_IF(X_s^I)]\,ds + \int_0^t P_IB(X_s^I)\,dW_s. \tag{115}$$

Then it holds for all $t \in [0,T]$ that $\mathbb{P}(X_t \in H_\eta) = 1$ and it holds that

$$\sup\nolimits_{R\in L^1(U)} \sup\nolimits_{I\in\mathcal{P}(\mathbb{H})} \sup\nolimits_{N\in\mathbb{N}} \sup\nolimits_{t\in[0,T]} \big(\|Y_t^{N,I,R}\|_{L^p(\mathbb{P};H_\eta)} + \|X_t^I\|_{L^p(\mathbb{P};H_\eta)}\big) < \infty. \tag{116}$$

Proof of Corollary 7.5. Combining Lemma 3.2 and Corollary 7.4 proves $\forall\, t \in [0,T]\colon P(X_t \in H_\eta) = 1$ and (116). The proof of Corollary 7.5 is thus completed. □

Theorem 7.6. *Assume the setting in Section 7.1, let $\nu \in (0, 1/2 - \delta)$, $\eta \in [\max\{\delta,\gamma\}, 1/2)$, $\kappa \in (2/p, \infty)$, assume that $\xi(\Omega) \subseteq H_\eta$, assume that $\mathbb{E}\big[\|\xi\|_{H_\eta}^{2a(c+1)p\max\{\kappa,1/\theta\}}\big] < \infty$, and assume that for all $x, y \in H_1$ it holds that $\langle x, F(x)\rangle_H + \frac{2a(c+1)p\max\{\kappa,1/\theta\}-1}{2}\|B(x)\|^2_{\mathrm{HS}(U,H)} \leq C(1+\|x\|^2_H)$ and $\langle x-y, Ax - Ay + F(x) - F(y)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|B(x) - B(y)\|^2_{\mathrm{HS}(U,H)} \leq C\,\|x-y\|^2_H$. Then there exists a real number $K \in [0,\infty)$ such that for all $N \in \mathbb{N}, I \in \mathcal{P}_0(\mathbb{H})$, $R \in L^1(U)$ it holds that*

$$\sup_{t\in[0,T]} \|X_t - Y_t^{N,I,R}\|_{L^p(\mathbb{P};H)} \leq K\left[N^{\delta-\eta} + \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{\delta-\eta})} + \sup_{v\in H_\eta}\Big(\tfrac{\|B(v)(\mathrm{Id}_U - R)\|_{\mathrm{HS}(U,H_{-\nu})}}{(1+\|v\|_{H_\eta})^\kappa}\Big)\right]. \tag{117}$$

Proof of Theorem 7.6. First of all, observe that it is well known that the fact that the functions $P_I(H) \ni x \mapsto P_I(F(x)) \in P_I(H), I \in \mathcal{P}_0(\mathbb{H})$, and $P_I(H) \ni x \mapsto P_I(B(x))R \in \mathrm{HS}(P(U), P_I(H)), I \in \mathcal{P}_0(\mathbb{H}), R \in L^1(U)$, are locally Lipschitz continuous and the fact that $\forall\, x \in H_1\colon \langle x, F(x)\rangle_H + \frac{2a(c+1)p\max\{\kappa,1/\theta\}-1}{2}\|B(x)\|^2_{\mathrm{HS}(U,H)} \leq C(1+\|x\|^2_H)$ ensure that there exist $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes $X^{I,R}\colon [0,T]\times\Omega \to P_I(H_\gamma), I \in \mathcal{P}_0(\mathbb{H}), R \in L^1(U)$, with continuous sample paths such that for all $t \in [0,T], I \in \mathcal{P}_0(\mathbb{H}), R \in L^1(U)$ it holds $\mathbb{P}$-a.s. that

$$X_t^{I,R} = P_I\xi + \int_0^t [AX_s^{I,R} + P_IF(X_s^{I,R})]\,ds + \int_0^t P_IB(X_s^{I,R})R\,dW_s. \tag{118}$$

Moreover, note that the triangle inequality proves that for all $t \in [0,T], N \in \mathbb{N}, I, \tilde{I} \in \mathcal{P}_0(\mathbb{H})$, $R \in L^1(U)$ with $I \subseteq \tilde{I}$ it holds that

$$\begin{aligned}
\|X_t - Y_t^{N,I,R}\|_{L^p(\mathbb{P};H)} &\leq \|X_t - X_t^{\tilde{I},\mathrm{Id}_U}\|_{L^p(\mathbb{P};H)} + \|X_t^{\tilde{I},\mathrm{Id}_U} - X_t^{I,\mathrm{Id}_U}\|_{L^p(\mathbb{P};H)} \\
&\quad + \|X_t^{I,\mathrm{Id}_U} - X_t^{I,R}\|_{L^p(\mathbb{P};H)} + \|X_t^{I,R} - Y_t^{N,I,R}\|_{L^p(\mathbb{P};H)}.
\end{aligned} \tag{119}$$

In the next step we note that the assumption that H is separable implies that there exist non-decreasing sets $I_n \in \mathcal{P}_0(\mathbb{H})$, $n \in \mathbb{N}$, with the property that $\cup_{n\in\mathbb{N}} I_n = \mathbb{H}$. Next we combine Corollary 4.4, Lemma 7.1, and Lemma 7.2 to obtain that for all $t \in [0,T], N, n \in \mathbb{N}, I \in \mathcal{P}_0(\mathbb{H}), R \in L^1(U)$ with $I \subseteq I_n$ it holds that

$$\begin{aligned}
&\|X_t - Y_t^{N,I,R}\|_{L^p(\mathbb{P};H)} \leq \|X_t - X_t^{I_n,\mathrm{Id}_U}\|_{L^p(\mathbb{P};H)} + \big(\|(-A)^{-\delta}\|_{L(H)} + e^{(C+1)T}Cp(1+\tfrac{1}{\varepsilon})\big)\\
&\cdot \sup\nolimits_{u\in[0,T]} \|P_{\mathbb{H}\setminus I} X_u^{I_n,\mathrm{Id}_U}\|_{L^{2p}(\mathbb{P};H_\delta)} \big[1 + 2\sup\nolimits_{u\in[0,T]} \|X_u^{I_n,\mathrm{Id}_U}\|^c_{L^{2pc}(\mathbb{P};H_\gamma)}\big] + \tfrac{\sqrt{p(2p-1)}\max\{1,T^{1/2}\}}{\sqrt{1-2(\delta+\nu)}}\\
&\cdot \Big(1 + \exp\Big(\tfrac{TC^2p(1+1/\varepsilon)}{2}\Big)\big(TC^2p(1+\tfrac{1}{\varepsilon})\big)^{\frac{1}{p}}\Big)\Big(1 + \sup\nolimits_{s\in[0,T]} \|X_s^{I,\mathrm{Id}_U}\|^\kappa_{L^{2p\kappa}(\mathbb{P};H_\eta)}\Big)\\
&\cdot \Bigg[\sup\nolimits_{v\in H_\eta} \tfrac{\|B(v)(\mathrm{Id}_U - R)\|_{\mathrm{HS}(U,H_{-\nu})}}{1+\|v\|^\kappa_{H_\eta}}\Bigg]\Bigg[3\big(1 + \|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)}\big)C\max\{1,T\}\Bigg[\tfrac{1}{1-(\gamma+\alpha)} + \sqrt{\tfrac{pc(2pc-1)}{1-2(\gamma+\beta)}}\Bigg]\\
&\cdot \Big[1 + \sup\nolimits_{t\in[0,T]} \|X_t^{I,\mathrm{Id}_U}\|^a_{L^{2pca}(\mathbb{P};H)}\Big]\Bigg]^c + N^{\delta-\eta}\tfrac{\max\{1,T^2\}}{(1-2\eta)}\big(C^2(1+1/\varepsilon)p\big)^{1/p}\exp\Big(\tfrac{TC^2p(1+1/\varepsilon)}{2}\Big)\\
&\cdot \Bigg(3\Big[1 + \sup\nolimits_{s\in[0,T]} \|F(Y_s^{N,I,R})\|_{L^{2p/\theta}(\mathbb{P};H)} + \sqrt{p(2p-1)}\sup\nolimits_{s\in[0,T]} \|B(Y_s^{N,I,R})\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big]^{1+\frac{1}{2\theta}}\\
&+ \sup\nolimits_{s\in[0,T]} \|Y_s^{N,I,R}\|_{L^{2p}(\mathbb{P};H_\eta)}\Bigg)\Bigg(1 + 2\Bigg[\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C\Bigg[\tfrac{T^{1-(\gamma+\alpha)}}{1-(\gamma+\alpha)} + \sqrt{\tfrac{pc(2pc-1)}{(1-2(\gamma+\beta))}}T^{\frac{1}{2}-(\gamma+\beta)}\Bigg]\\
&\cdot \Big[1 + \sup\nolimits_{s\in[0,T]} \|Y_s^{N,I,R}\|^a_{L^{2pca}(\mathbb{P};H)}\Big]\Bigg]^c\Bigg).
\end{aligned} \tag{120}$$

Moreover, observe that Proposition 7.3 shows that

$$\sup_{I\in\mathcal{P}_0(\mathbb{H})}\ \sup_{R\in L^1(U)}\ \sup_{t\in[0,T]}\ \sup_{N\in\mathbb{N}} \Big(\|X_t\|_{L^{2pc}(\mathbb{P};H_\gamma)} + \|X_t^{I,\mathrm{Id}_U}\|_{L^{2pca}(\mathbb{P};H)} + \|Y_t^{N,I,R}\|_{L^{2pca}(\mathbb{P};H)}\Big) < \infty \tag{121}$$

and $\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}[\|X_t - X_t^{I_n,\mathrm{Id}_U}\|^{2pc}_{H_\gamma}] = 0$. Combining this with (120) implies that for all $N \in \mathbb{N}, I \in \mathcal{P}_0(\mathbb{H})$, $R \in L^1(U)$ it holds that

$$\begin{aligned}
&\sup\nolimits_{t\in[0,T]} \|X_t - Y_t^{N,I,R}\|_{L^p(\mathbb{P};H)} \leq \big(\|(-A)^{-\delta}\|_{L(H)} + e^{(C+1)T}Cp(1+\tfrac{1}{\varepsilon})\big)\sup\nolimits_{u\in[0,T]} \|P_{\mathbb{H}\setminus I} X_u\|_{L^{2p}(\mathbb{P};H_\delta)}\\
&\cdot\big[1 + 2\sup\nolimits_{u\in[0,T]} \|X_u\|^c_{L^{2pc}(\mathbb{P};H_\gamma)}\big] + \tfrac{\sqrt{p(2p-1)}\max\{1,T^{1/2}\}}{\sqrt{1-2(\delta+\nu)}}\Bigg[\sup\nolimits_{v\in H_\eta} \tfrac{\|B(v)(\mathrm{Id}_U - R)\|_{\mathrm{HS}(U,H_{-\nu})}}{1+\|v\|^\kappa_{H_\eta}}\Bigg]\\
&\cdot \Big(1 + \exp\Big(\tfrac{TC^2p(1+1/\varepsilon)}{2}\Big)\big(TC^2p(1+\tfrac{1}{\varepsilon})\big)^{\frac{1}{p}}\Big)\Big(1 + \sup\nolimits_{s\in[0,T]} \|X_s^{I,\mathrm{Id}_U}\|^\kappa_{L^{2p\kappa}(\mathbb{P};H_\eta)}\Big)\\
&\cdot \Bigg[3\big(1 + \|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)}\big)C\max\{1,T\}\Bigg[\tfrac{1}{1-(\gamma+\alpha)} + \sqrt{\tfrac{pc(2pc-1)}{1-2(\gamma+\beta)}}\Bigg]\big[1 + \sup\nolimits_{t\in[0,T]} \|X_t^{I,\mathrm{Id}_U}\|^a_{L^{2pca}(\mathbb{P};H)}\big]\Bigg]^c\\
&+ N^{\delta-\eta}\tfrac{\max\{1,T^2\}}{(1-2\eta)}\big(C^2(1+1/\varepsilon)p\big)^{1/p}\exp\Big(\tfrac{TC^2p(1+1/\varepsilon)}{2}\Big)\Bigg(3\Big[1 + \sup\nolimits_{s\in[0,T]} \|F(Y_s^{N,I,R})\|_{L^{2p/\theta}(\mathbb{P};H)}\\
&+ \sqrt{p(2p-1)}\sup\nolimits_{s\in[0,T]} \|B(Y_s^{N,I,R})\|_{L^{2p/\theta}(\mathbb{P};\mathrm{HS}(U,H))}\Big]^{1+\frac{1}{2\theta}} + \sup\nolimits_{s\in[0,T]} \|Y_s^{N,I,R}\|_{L^{2p}(\mathbb{P};H_\eta)}\Bigg)\\
&\cdot \Bigg(1 + 2\Bigg[\|\xi\|_{L^{2pc}(\mathbb{P};H_\gamma)} + C\Bigg[\tfrac{T^{1-(\gamma+\alpha)}}{1-(\gamma+\alpha)} + \sqrt{\tfrac{pc(2pc-1)}{(1-2(\gamma+\beta))}}T^{\frac{1}{2}-(\gamma+\beta)}\Bigg]\Big[1 + \sup\nolimits_{s\in[0,T]} \|Y_s^{N,I,R}\|^a_{L^{2pca}(\mathbb{P};H)}\Big]\Bigg]^c\Bigg).
\end{aligned} \tag{122}$$

Furthermore, note that Corollary 7.5 proves that for all $t \in [0,T]$ it holds that $\mathbb{P}(X_t \in H_\eta) = 1$ and

$$\sup_{I\in\mathcal{P}_0(\mathbb{H})}\ \sup_{R\in L^1(U)}\ \sup_{t\in[0,T]}\ \sup_{N\in\mathbb{N}} \Big(\|Y_t^{N,I,R}\|_{L^{2p}(\mathbb{P};H_\eta)} + \|X_t^{I,\mathrm{Id}_U}\|_{L^{2p\kappa}(\mathbb{P};H_\eta)} + \|X_t\|_{L^{2p}(\mathbb{P};H_\eta)}\Big) < \infty. \tag{123}$$

Combining Corollary 7.4 and (121)–(123) proves that there exists a real number $\tilde{K} \in [0,\infty)$ such that for all $N \in \mathbb{N}, I \in \mathcal{P}_0(\mathbb{H}), R \in L^1(U)$ it holds that

$$\sup_{t\in[0,T]} \|X_t - Y_t^{N,I,R}\|_{L^p(\mathbb{P};H)} \leq \tilde{K}\left[\sup_{t\in[0,T]} \|P_{\mathbb{H}\setminus I} X_t\|_{L^{2p}(\mathbb{P};H_\delta)} + N^{\delta-\eta} + \sup_{v\in H_\eta}\left(\tfrac{\|B(v)(\mathrm{Id}_U - R)\|_{\mathrm{HS}(U,H_{-\nu})}}{1+\|v\|_{H_\eta}^{\kappa}}\right)\right]. \tag{124}$$

Moreover, note that for all $I \in \mathcal{P}_0(\mathbb{H})$ it holds that

$$\sup_{t\in[0,T]} \|P_{\mathbb{H}\setminus I} X_t\|_{L^{2p}(\mathbb{P};H_\delta)} \leq \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{\delta-\eta})} \sup_{t\in[0,T]} \|X_t\|_{L^{2p}(\mathbb{P};H_\eta)}. \tag{125}$$

Combining this, (123), and (124) completes the proof of Theorem 7.6. □

8 A stochastic reaction-diffusion partial differential equation

In this section we illustrate Theorem 7.6 by a simple example, that is, we illustrate Theorem 7.6 in the case of a stochastic reaction-diffusion partial differential equation. More formally, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0,1)$ to $\mathbb{R}$, let $T,\rho,\kappa,\varepsilon,\sigma \in (0,\infty)$, $\theta \in (0, 1/4]$, $\gamma \in (1/4, 1/2)$, $(e_n)_{n\in\mathbb{N}} \subseteq H$, $(r_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy that for all $n \in \mathbb{N}$ and $\mu_{(0,1)}$-almost all $x \in (0,1)$ it holds that $e_n(x) = \sqrt{2}\sin(n\pi x)$, $\lambda_n = -\varepsilon\pi^2 n^2$, and $\sup_{m\in\mathbb{N}}(m \cdot |r_m|) < \infty$, let $A\colon D(A) \subseteq H \to H$ be the linear operator such that $D(A) = \big\{v \in H\colon \sum_{n=1}^\infty |\lambda_n \langle e_n, v\rangle_H|^2 < \infty\big\}$ and such that for all $v \in D(A)$ it holds that $Av = \sum_{n=1}^\infty \lambda_n \langle e_n, v\rangle_H e_n$, let $Q \in L_1(H)$ be the linear operator such that for all $v \in H$ it holds that $Qv = \sum_{n=1}^\infty r_n \langle e_n, v\rangle_H e_n$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (see, e.g., Theorem and Definition 2.5.32 in [16]), let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $\xi \in H_{1/2}$ satisfy that for $\mu_{(0,1)}$-almost all $x \in (0,1)$ it holds that $\xi(x) \geq 0$, let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_H-Wiener process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, let $F \in \mathcal{C}(H_\gamma, H)$ and $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(H))$ be the functions with the properties that for all $v \in H_\gamma$, $u \in H$ and $\mu_{(0,1)}$-almost all $x \in (0,1)$ it holds that $\big(F(v)\big)(x) = \kappa\,|v(x)|\,(\rho - v(x))$ and $\big(B(v)(u)\big)(x) = \sigma \cdot v(x) \cdot \big(\sqrt{Q}u\big)(x)$, let $(P_n)_{n\in\mathbb{N}} \subseteq L(H)$ be the linear operators with the property that for all $x \in H$, $n \in \mathbb{N}$ it holds that $P_n(x) = \sum_{l=1}^n \langle e_l, x\rangle_H e_l$, let $Y^{N,n,m}\colon [0,T]\times\Omega \to P_n(H_\gamma)$, $N,n,m,\in \mathbb{N}$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes such that for all $t \in [0,T]$, $N,n,m \in \mathbb{N}$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} Y_t^{N,n,m} &= e^{tA}P_n\xi + \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\,\mathbb{1}_{\left\{\|P_nF(Y^{N,n,m}_{\lfloor s\rfloor_{T/N}})\|_H + \|P_nB(Y^{N,n,m}_{\lfloor s\rfloor_{T/N}})\|_{HS(H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} P_nF(Y^{N,n,m}_{\lfloor s\rfloor_{T/N}})\,ds \\ &\quad + \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\,\mathbb{1}_{\left\{\|P_nF(Y^{N,n,m}_{\lfloor s\rfloor_{T/N}})\|_H + \|P_nB(Y^{N,n,m}_{\lfloor s\rfloor_{T/N}})\|_{HS(H)} \leq \left(\frac{N}{T}\right)^\theta\right\}} P_nB(Y^{N,n,m}_{\lfloor s\rfloor_{T/N}})P_m\,dW_s, \end{aligned} \tag{126}$$

and let $X\colon [0,T]\times\Omega \to H_\gamma$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A}F(X_s)\,ds + \int_0^t e^{(t-s)A}B(X_s)\,dW_s. \tag{127}$$

The stochastic process X is thus a solution process of the SPDE

$$dX_t(x) = \left[\varepsilon\tfrac{\partial^2}{\partial x^2}X_t(x) + \kappa X_t(x)\,(\rho - X_t(x))\right]dt + \sigma X_t(x)\,dW_t(x),\; X_0(x) = \xi(x),\; X_t(0) = X_t(1) = 0$$

for $t \in [0,T]$, $x \in (0,1)$. We intend to apply Theorem 7.6 to estimate the quantities $\sup_{t\in[0,T]} \|X_t - Y_t^{N,n,m}\|_{L^p(\mathbb{P};H)}$ for $p \in [2,\infty)$, $N,n,m \in \mathbb{N}$. To this end, we now check the assumptions of Theorem 7.6. First, observe that the assumption that $\gamma > 1/4$ ensures that for all $v \in H_\gamma$ it holds that

$$
\begin{aligned}
\|F(v)\|_{H_{-\gamma}} &= \sup_{z\in H_\gamma\setminus\{0\}} \left(\tfrac{|\langle F(v),z\rangle_H|}{\|z\|_{H_\gamma}}\right) = \sup_{z\in H_\gamma\setminus\{0\}} \left(\tfrac{1}{\|z\|_{H_\gamma}} \left|\int_0^1 \kappa\cdot|v(x)|\cdot(\rho - v(x))\cdot z(x)\,dx\right|\right) \\
&\leq \sup_{z\in H_\gamma\setminus\{0\}} \left(\tfrac{\|z\|_{L^\infty(\mu_{(0,1)};\mathbb{R})}}{\|z\|_{H_\gamma}} \int_0^1 \kappa\cdot|v(x)|\cdot|\rho - v(x)|\,dx\right) \\
&\leq \kappa \left(\sup_{z\in H_\gamma\setminus\{0\}} \tfrac{\|z\|_{L^\infty(\mu_{(0,1)};\mathbb{R})}}{\|z\|_{H_\gamma}}\right) \|v\|_H \left(\rho + \|v\|_H\right) \\
&\leq \left(\sup_{z\in H_\gamma\setminus\{0\}} \tfrac{\|z\|_{L^\infty(\mu_{(0,1)};\mathbb{R})}}{\|z\|_{H_\gamma}}\right) \tfrac{3\,\kappa\max\{1,\rho\}}{2} \left(1 + \|v\|_H^2\right) < \infty.
\end{aligned}
\tag{128}
$$

Next note that the triangle inequality shows that for all $x,y \in \mathbb{R}$ it holds that

$$
\begin{aligned}
&\left||x|\,(\rho - x) - |y|\,(\rho - y)\right| \leq \rho\,|x-y| + \left||x|\,x - |y|\,y\right| \\
&\leq \rho\,|x-y| + \left||x|\,x - |y|\,x\right| + \left||y|\,x - |y|\,y\right| \leq \rho\,|x-y| + |x|\,\left||x| - |y|\right| + |y|\,|x-y| \\
&\leq \rho\,|x-y| + |x|\,|x-y| + |y|\,|x-y| \leq \max\{1,\rho\}\,|x-y|\,(1 + |x| + |y|)\,.
\end{aligned}
\tag{129}
$$

This implies that for all $u,v \in H_\gamma$ it holds that

$$
\begin{aligned}
\|F(u) - F(v)\|_H &= \kappa \left(\int_0^1 \left||u(s)|\,(\rho - u(s)) - |v(s)|\,(\rho - v(s))\right|^2 ds\right)^{1/2} \\
&\leq \kappa\max\{1,\rho\} \left(\int_0^1 |u(s) - v(s)|^2\,[1 + |u(s)| + |v(s)|]^2\,ds\right)^{1/2} \\
&\leq \left(\sup_{z\in H_\gamma\setminus\{0\}} \tfrac{\|z\|_{L^\infty(\mu_{(0,1)};\mathbb{R})}}{\|z\|_{H_\gamma}}\right) \kappa\max\{1,\rho\}\,\|u - v\|_H \left(1 + \|u\|_{H_\gamma} + \|v\|_{H_\gamma}\right).
\end{aligned}
\tag{130}
$$

Moreover, note that for all $u,v \in H_\gamma$ it holds that

$$
\begin{aligned}
\|B(u) - B(v)\|_{\mathrm{HS}(H)} = \|B(u-v)\|_{\mathrm{HS}(H)} &= \sigma \left[\sum_{n=1}^\infty \left\|(u-v)\,(Q^{1/2}e_n)\right\|_H^2\right]^{1/2} \\
&= \sigma \left[\sum_{n=1}^\infty r_n\,\|(u-v)\,e_n\|_H^2\right]^{1/2} \leq \sigma\sqrt{2\,\mathrm{Trace}(Q)}\,\|u-v\|_H\,.
\end{aligned}
\tag{131}
$$

This shows that for all $v \in H_\gamma$, $p \in [0,\infty)$ it holds that

$$
\langle v, F(v)\rangle_H + p\,\|B(v)\|_{\mathrm{HS}(H)}^2 \leq \kappa\rho\,\|v\|_H^2 + p\,\|B(v)\|_{\mathrm{HS}(H)}^2 \leq \left(\kappa\rho + 2p\,|\sigma|^2\,\mathrm{Trace}(Q)\right)\|v\|_H^2\,.
\tag{132}
$$

Furthermore, note that (131) and the fact that the function $\mathbb{R} \ni x \mapsto -x\,|x| \in \mathbb{R}$ is non-decreasing show that for all $u,v \in H_1$, $p \in [0,\infty)$ it holds that

$$
\begin{aligned}
&\langle u - v, Au - Av + F(u) - F(v)\rangle_H + p\,\|B(u) - B(v)\|_{\mathrm{HS}(H)}^2 \\
&\leq \kappa\rho\,\|u-v\|_H^2 - \kappa\,\langle u - v, u\,|u| - v\,|v|\rangle_H + 2p\,|\sigma|^2\,\mathrm{Trace}(Q)\,\|u-v\|_H^2 \\
&\leq \left(\kappa\rho + 2p\,|\sigma|^2\,\mathrm{Trace}(Q)\right)\|u-v\|_H^2\,.
\end{aligned}
\tag{133}
$$

Combining (128), (130), (131), (132), and (133) allows us to apply Theorem 7.6 to obtain that for every $\eta \in [\gamma, 1/2)$, $\nu \in (0, 1/2)$, $\kappa \in (0, \infty)$, $p \in [2, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $N, n, m \in \mathbb{N}$ it holds that

$$\sup_{t \in [0,T]} \left\| X_t - Y_t^{N,n,m} \right\|_{L^p(\mathbb{P};H)} \leq K \left[N^{-\eta} + n^{-2\eta} + \sup_{v \in H_\eta} \left(\tfrac{\|B(v)(\mathrm{Id}_H - P_m)\|_{\mathrm{HS}(H, H_{-\nu})}}{(1 + \|v\|_{H_\eta})^\kappa} \right) \right]. \tag{134}$$

In the next step we intend to estimate the third summand on the right hand side of (134). For this let $\|\cdot\|_{H^r((0,1),\mathbb{R})} \colon H \to [0, \infty]$, $r \in (0, 1)$, be the functions with the property that for all $r \in (0, 1)$, $v \in H$ it holds that

$$\|v\|_{H^r((0,1),\mathbb{R})} = \left[\int_0^1 |v(x)|^2 \, dx + \int_0^1 \int_0^1 \tfrac{|v(x) - v(y)|^2}{|x-y|^{1+2r}} \, dx \, dy \right]^{1/2}. \tag{135}$$

Note that there exists real numbers $\vartheta_r \in [1, \infty)$, $r \in (0, 1/2)$, such that for all $r \in (0, 1/2)$, $v \in H_r$ it holds that

$$\tfrac{1}{\vartheta_r} \|v\|_{H^{2r}((0,1),\mathbb{R})} \leq \|v\|_{H_r} \leq \vartheta_r \|v\|_{H^{2r}((0,1),\mathbb{R})} \tag{136}$$

(see, e.g., A. Lunardi [23] or also (A.46) in Da Prato & Zabczyk [5]). In addition, we observe that for all $u, v \in \mathcal{M}(\mathcal{B}((0,1)), \mathcal{B}(\mathbb{R}))$, $r \in (0, 1)$, $s \in (r, \infty)$ it holds that

$$\|u \cdot v\|_{H^r((0,1),\mathbb{R})} \leq \sqrt{2} \, \|v\|_{H^r((0,1),\mathbb{R})} \left(\sup\nolimits_{x \in (0,1)} |u(x)| + \tfrac{\sqrt{3}}{\sqrt{2s - 2r}} \sup\nolimits_{x \in (0,1), y \in (x,1)} \tfrac{|u(x) - u(y)|}{|x-y|^s} \right) \tag{137}$$

(cf., e.g., Jentzen & Röckner [19, (22)–(23)]). This and (136) prove that for all $m \in \mathbb{N}$, $r \in [\gamma, 1/2)$, $s \in (2r - 1/2, \infty)$, $\nu \in (s/2 + 1/4, \infty)$, $v \in H_r$ it holds that

$$
\begin{aligned}
&\|B(v)(\mathrm{Id}_H - P_m)\|_{\mathrm{HS}(H,H_{-\nu})} = \left\|(-A)^{-\nu} B(v)(\mathrm{Id}_H - P_m)\right\|_{\mathrm{HS}(H)} \\
&= \|(\mathrm{Id}_H - P_m) B(v)^* (-A)^{\nu}\|_{\mathrm{HS}(H)} \leq \left\|(\mathrm{Id}_H - P_m)(-A)^{-r}\right\|_{L(H)} \left\|(-A)^r B(v)^*(-A)^{-\nu}\right\|_{\mathrm{HS}(H)} \\
&= |\lambda_{m+1}|^{-r} \left\|(-A)^r B(v)^* (-A)^{-\nu}\right\|_{\mathrm{HS}(H)} = |\lambda_m|^{-r} \left[\sum_{n=1}^{\infty} \left\|(-A)^r B(v)^*(-A)^{-\nu} e_n\right\|_H^2\right]^{1/2} \\
&= \frac{1}{|\lambda_m|^r} \left[\sum_{n,k=1}^{\infty} |r_k| \, |\lambda_n|^{-2\nu} \left|\langle e_k, (-A)^r (v \cdot e_n)\rangle_H\right|^2\right]^{1/2} \\
&\leq \frac{1}{|\lambda_m|^r} \left[\sup_{k \in \mathbb{N}} |r_k| \, (\lambda_k)^{1/2}\right]^{1/2} \left[\sum_{n,k=1}^{\infty} |\lambda_n|^{-2\nu} \left|\left\langle e_k, (-A)^{(r-1/4)}(v \cdot e_n)\right\rangle_H\right|^2\right]^{1/2} \\
&\leq \frac{\vartheta_{r-1/4}}{|\lambda_m|^r} \left[\sup_{k \in \mathbb{N}} |r_k|^{1/2} (\lambda_k)^{1/4}\right] \left[\sum_{n=1}^{\infty} |\lambda_n|^{-2\nu} \, \|v \cdot e_n\|_{H^{2r-1/2}}^2\right]^{1/2} \qquad (138) \\
&\leq \frac{\vartheta_{r-1/4} \sqrt{2} \, \|(-A)^{1/4} Q^{1/2}\|_{L(H)} \, \|v\|_{H^{2r-1/2}((0,1),\mathbb{R})}}{|\lambda_m|^r} \left[\sum_{n=1}^{\infty} \frac{\left[\sqrt{2} + \frac{\sqrt{3}}{\sqrt{2s-4r}} \sup_{x \in (0,1), y \in (x,1)} \frac{|e_n(x) - e_n(y)|}{|x-y|^s}\right]^2}{|\lambda_n|^{2\nu}}\right]^{1/2} \\
&= \frac{2 \, \vartheta_{r-1/4} \, \|(-A)^{1/4} Q^{1/2}\|_{L(H)} \, \|v\|_{H^{2r-1/2}((0,1),\mathbb{R})}}{|\lambda_m|^r} \left[\sum_{n=1}^{\infty} \frac{\left[1 + \frac{\sqrt{3}}{\sqrt{2s-4r}} \sup_{x \in (0,1)} \sup_{y \in (x,1)} \frac{|\sin(n\pi x) - \sin(n\pi y)|^s}{|x-y|^s}\right]^2}{|\lambda_n|^{2\nu}}\right]^{1/2} \\
&\leq \frac{2 \left|\vartheta_{r-1/4}\right|^2 \|(-A)^{1/4} Q^{1/2}\|_{L(H)} \, \|v\|_{H_{r-1/4}}}{\varepsilon^{\nu} \, \pi^{2\nu} \, |\lambda_m|^r} \left[\sum_{n=1}^{\infty} n^{-4\nu} \left[1 + \tfrac{\sqrt{3}}{\sqrt{1+2s-4r}} \, (n\pi)^s\right]^2\right]^{1/2} \\
&\leq \frac{2 \left|\vartheta_{r-1/4}\right|^2 \|(-A)^{1/4} Q^{1/2}\|_{L(H)} \, \|v\|_{H_{r-1/4}}}{\varepsilon^{(\nu+r)} \, \pi^{(2\nu+2r)} \, m^{2r}} \left[1 + \tfrac{\pi^s \sqrt{3}}{\sqrt{1+2s-4r}}\right] \left[\sum_{n=1}^{\infty} n^{(2s-4\nu)}\right] < \infty.
\end{aligned}
$$

This implies that for all $m \in \mathbb{N}$, $\eta \in [\gamma, 1/2)$, $s \in (2\eta - 1/2, 1/2)$, $\nu \in (s/2 + 1/4, 1/2)$ it holds that

$$
\sup_{v \in H_\eta} \left(\frac{\|B(v)(\mathrm{Id}_H - P_m)\|_{\mathrm{HS}(H,H_{-\nu})}}{1 + \|v\|_{H_\eta}}\right) \leq \frac{|\vartheta_{\eta-1/4}|^2 \, \|(-A)^{1/4} Q^{1/2}\|_{L(H)}}{\varepsilon^{(\nu+\eta+1/4)} \, m^{2\eta} \, (s/2 + 1/4 - \eta)} \left[\sum_{n=1}^{\infty} n^{(2s-4\nu)}\right] < \infty. \qquad (139)
$$

Combining this with (134) shows that for every $\eta \in [\gamma, 1/2)$, $p \in [2, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $N, n, m \in \mathbb{N}$ it holds that

$$
\sup_{t \in [0,T]} \left\|X_t - Y_t^{N,n,m}\right\|_{L^p(\mathbb{P};H)} \leq K \left[N^{-\eta} + n^{-2\eta} + m^{-2\eta}\right]. \qquad (140)
$$

This ensures that for every $p, \iota \in (0, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $N, n, m \in \mathbb{N}$ it holds that

$$
\sup_{t \in [0,T]} \left\|X_t - Y_t^{N,n,m}\right\|_{L^p(\mathbb{P};H)} \leq K \left(\frac{1}{N^{(1/2-\iota)}} + \frac{1}{n^{(1-\iota)}} + \frac{1}{m^{(1-\iota)}}\right). \qquad (141)
$$

In particular, this shows that for every $p, \iota \in (0, \infty)$ there exists a real number $K \in [0, \infty)$ such that for all $n \in \mathbb{N}$ it holds that $\sup_{t \in [0,T]} \|X_t - Y_t^{n^2,n,n}\|_{L^p(\mathbb{P};H)} \leq K \cdot n^{(\iota-1)}$.

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